# The Symplectic Sum Formula for Gromov-Witten Invariants

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#### Abstract

In the symplectic category there is a 'connect sum' operation that glues symplectic manifolds by identifying neighborhoods of embedded codimension two submanifolds. This paper establishes a formula for the Gromov-Witten invariants of a symplectic sum Z = X # Y in terms of the relative GW invariants of X and Y. Several applications to enumerative geometry are given.

Gromov-Witten invariants are counts of holomorphic maps into symplectic manifolds. To define them on a symplectic manifold  $(X,\omega)$  one introduces an almost complex structure J compatible with the symplectic form  $\omega$  and forms the moduli space of J-holomorphic maps from complex curves into X and its compactification, called the space of stable maps. One then imposes constraints on the stable maps, requiring the domain to have a certain form and the image to pass through fixed homology cycles in X. When the right number of constraints are imposed there are only finitely many maps satisfying the constraints; the (oriented) count of these is the corresponding GW invariant. For complex algebraic manifolds these symplectic invariants can also be defined by algebraic geometry, and in important cases the invariants are the same as the curve counts that are the subject of classical enumerative algebraic geometry.

In the past decade the foundations for this theory were laid and the invariants were used to solve several long-outstanding problems. The focus now is on finding effective ways of computing the invariants. One useful technique is the method of 'splitting the domain', in which one localizes the invariant to the set of maps whose domain curves have two irreducible components with the constraints distributed between them. This produces recursion relations relating the desired GW invariant to invariants with lower degree or genus. This paper establishes a general formula describing the behavior of GW invariants under the analogous operation of 'splitting the target'. Because we work in the context of symplectic manifolds the natural splitting of the target is the one associated with the symplectic cut operation and its inverse, the symplectic sum.

The symplectic sum is defined by gluing along codimension two submanifolds. Specifically, let X be a symplectic 2n-manifold with a symplectic (2n-2)-submanifold V. Given a similar

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pair (Y, V) with a symplectic identification between the 2 copies of V and a complex antilinear isomorphism between the normal bundles  $N_X$  and  $N_Y$  of V in X and in Y we can form the symplectic sum  $Z = X \#_V Y$ . Our main theorem is a 'Symplectic Sum Formula' which expresses the GW invariants of the sum Z in terms of relative GW invariants of (X, V) and (Y, V) introduced in [IP4].

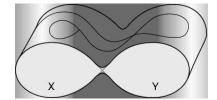
The symplectic sum is perhaps more naturally seen not as a single manifold but as a family depending on a 'squeezing parameter'. In Section 2 we construct a family  $Z \to D$  over the disk whose fibers  $Z_{\lambda}$  are smooth and symplectic for  $\lambda \neq 0$  and whose central fiber  $Z_0$  is the singular manifold  $X \cup_V Y$ . In a neighborhood of V, the total space Z is  $N_X \oplus N_Y$ , regarded as a subset of  $X \times Y$  by the symplectic neighborhood theorem, and the fiber  $Z_{\lambda}$  is defined by the equation  $xy = \lambda$  where x and y are coordinates in the normal bundles  $N_X$  and  $N_Y \cong N_X^*$ . The fibration  $Z \to D$  extends away from V as the disjoint union of  $X \times D$  and  $Y \times D$ . The smooth fibers  $Z_{\lambda}$ , depicted in Figure 1, are symplectically isotopic to one another; each is a model of the symplectic sum.

The overall strategy for proving the symplectic sum formula is to relate the holomorphic maps into  $Z_0$  (which are simply maps into X and Y which match along V) with the holomorphic maps into  $Z_{\lambda}$  for  $\lambda$  close to zero. This strategy involves two parts: limits and gluing. For the limiting process we consider sequences of stable maps into the family  $Z_{\lambda}$  of symplectic sums as the 'neck size'  $\lambda \to 0$ . In Section 3 we show that these limit to maps into the singular manifold  $Z_0$  obtained by identifying X and Y along V. Along the way several things become apparent.

First, the limit maps are holomorphic only if the almost complex structures on X and Y match along V. To ensure that we impose the "V-compatibility" condition (1.10) on the almost complex structure. There is a price to pay for that. In the symplectic theory of Gromov-Witten invariants we are free to perturb  $(J, \nu)$  without changing the invariant; that freedom can be used to ensure that intersections are transverse. After imposing the V-compatibility condition we can no longer perturb  $(J, \nu)$  along V at will, and hence we cannot assume that the limit curves are transverse to V. In fact, the components of the limit maps meet V at points with multiplicities and, worse, some components may lie entirely in V.

To count such maps into  $Z_0$  we look first on the X side, ignore the maps with components in V, separate the moduli space of stable maps into components  $\mathcal{M}_s(X)$  labeled by the multiplicities  $s = (s_1, \ldots, s_\ell)$  of their intersection points with V. We showed in [IP4] how these spaces  $\mathcal{M}_s(X)$  can be compactified and used to define relative Gromov-Witten invariants  $GW_X^V$ . The definitions are briefly reviewed in Section 1.

Figure 1: Limiting curves in  $Z_{\lambda} = X \#_{\lambda} Y$  as  $\lambda \to 0$ .



Second, as Figure 1 illustrates, connected curves in  $Z_{\lambda}$  can limit to curves whose restrictions to X and Y are not connected. For that reason the GW invariant, which counts stable curves from a connected domain, is not the appropriate invariant for expressing a sum formula. Instead one should work with the 'Taubes-Witten' invariant TW, which counts stable maps from domains

that need not be connected. Thus we seek a formula of the general form

$$TW_X^V * TW_Y^V = TW_Z (0.1)$$

where \* is some operation that adds up the ways curves on the X and Y sides match and are identified with curves in  $Z_{\lambda}$ . That necessarily involves keeping track of the multiplicities s and the homology classes. It also involves accounting for the limit maps with non-trivial components in V; such curves are not counted by the relative invariant and hence do not contribute to the left side of (0.1). We postpone this issue by first analyzing limits of curves which are  $\delta$ -flat in the sense of Definition 3.1.

A more precise analysis reveals a third complication: the squeezing process is not injective. In Section 5 we again consider a sequence of stable maps  $f_n$  into  $Z_{\lambda}$  as  $\lambda \to 0$ , this time focusing on their behavior near V, where the  $f_n$  do not uniformly converge. We form renormalized maps  $\hat{f}_n$  and prove that both the domains and the images of the renormalized maps converge. The images converge nicely according to the leading order term of their Taylor expansions, but the domains converge only after fixing certain roots of unity.

These roots of unity are apparent as soon as one writes down formulas. Each stable map  $f:C\to Z_0$  decomposes into a pair of maps  $f_1:C_1\to X$  and  $f_2:C_2\to Y$  which agree at the nodes of  $C=C_1\cup C_2$ . For a specific example, suppose that f is such a map that intersects V at a single point p with multiplicity three. Then we can choose local coordinates z on  $C_1$  and w on  $C_2$  centered at the node, and coordinates x on X and y on Y so that  $f_1$  and  $f_2$  have expansions  $x(z)=az^3+\cdots$  and  $y(w)=bw^3+\cdots$ . To find maps into  $Z_\lambda$  near f, we smooth the domain C to the curve  $C_\mu$  given locally near the node by  $zw=\mu$  and require that the image of the smoothed map lie in  $Z_\lambda$ , which is locally the locus of  $xy=\lambda$ . In fact, the leading terms in the formulas for  $f_1$  and  $f_2$  define a map  $F:C_\mu\to Z_\lambda$  whenever

$$\lambda = xy = az^3 \cdot bw^3 = ab(zw)^3 = ab\mu^3$$

and conversely any family of smooth maps with limit to f satisfy this equation in the limit (c.f. Lemma 5.4). Thus  $\lambda$  determines the domain  $C_{\mu}$  up to a cube root of unity. That means that this particular f is, at least a priori, close to three smooth maps into  $Z_{\lambda}$  — a 'cluster' of order three.

Other maps f into  $Z_0$  have larger associated clusters (the order of the cluster is the product of the multiplicities with which f intersects V). Within a cluster, the maps have the same leading order formula but have different smoothings of the domain. As  $\lambda \to 0$  the maps within the cluster coalesce, limiting to the single map f.

This clustering phenomenon greatly complicates the analysis. To distinguish the curves within each cluster and make the analysis uniform in  $\lambda$  as  $\lambda \to 0$  it is necessary to use 'rescaled' norms and distances which magnify distances as the clusters form. With the right choice of norms, the distances between the maps within a cluster are bounded away from zero as  $\lambda \to 0$  and become the fiber of a covering of the space of limit maps. Sections 4– 6 introduce the required norms, first on the space of curves, then on the space of maps.

For maps we use a Sobolev norm weighted in the directions perpendicular to V; the weights are chosen so the norm dominates the  $C^0$  distance between the renormalized maps  $\hat{f}$ . On the space of curves we require a stronger metric than the usual complete metrics on  $\overline{\mathcal{M}}_{g,n}$ . In section 4 we define a complete metric on  $\overline{\mathcal{M}}_{g,n} \setminus \mathcal{N}$  where  $\mathcal{N}$  is the set of all nodal curves. In this metric the distance between two sequences that approach  $\mathcal{N}$  from different directions (corresponding to

the roots of unity mentioned above) is bounded away from zero; thus this metric separates the domain curves of maps within a cluster. The metric leads to a compactification of  $\overline{\mathcal{M}}_{g,n} \setminus \mathcal{N}$  in which the stratum  $\mathcal{N}_{\ell}$  of  $\ell$ -nodal curves is replaced by a bundle over  $\mathcal{N}_{\ell}$  whose fiber is the real torus  $T^{\ell}$ .

The limit process is reversed by constructing a space of approximately holomorphic maps and showing it is diffeomorphic to the space of stable maps into  $Z_{\lambda}$ . The space of approximate maps is described in Section 6, first intrinsically, then as a subset  $\mathcal{AM}_s$  of the space of maps. For each s and  $\lambda$  it is a covering of the space  $\mathcal{M}_s(Z_0)$  of the  $\delta$ -flat maps into  $Z_0$  that meet V at points with multiplicities s. The fibers of this covering are the clusters – they are distinct maps into  $Z_{\lambda}$  which converge to the same limit as  $\lambda \to 0$ .

From there the analysis follows the standard technique that goes back to Taubes and Donaldson: correct the approximate maps to true holomorphic maps by constructing a partial right inverse to the linearization D and applying a fixed point theorem. That involves (a) showing that the operator  $D^*D$  is uniformly invertible as  $\lambda \to 0$ , and (b) proving a priori that every solution is close to an approximate solution, close enough to be in the domain of the fixed point theorem. Proposition 9.4 shows that (b) follows from the renormalization analysis of Section 4. But the eigenvalue estimate (a) proves to be surprisingly delicate and seems to succeed only with a very specific choice of norms.

The difficulty, of course, is that  $Z_{\lambda}$  becomes singular along V as  $\lambda \to 0$ . However, for small  $\lambda$  the bisectional curvature in the neck region is negative; a Bochner formula then shows that eigenfunctions with small eigenvalue cannot be concentrating in the neck. One can then reason that since the cokernel of D vanishes on  $Z_0$  (for generic J) it should also vanish on  $Z_{\lambda}$  for small  $\lambda$ . We make that reasoning rigorous by introducing exponential weight functions into the norms, thereby making the linearizations  $D_{\lambda}$  a continuous family of Fredholm maps. That in turn necessities further work on the Bochner formula, bounding the additional term that arises from the derivative of the weight functions. These estimates are carried out in Section 8.

The upshot of the analysis is a diffeomorphism between the approximate moduli space and the true moduli spaces

$$\mathcal{AM}_s(Z_\lambda) \stackrel{\cong}{\longrightarrow} \mathcal{M}_s(Z_\lambda)$$

which intertwines with the attaching map of the domains and the evaluation map into the target (Theorem 10.1). We then pass to homology, comparing and keeping track of the homology classes of the maps, the domains, and the constraints. This involves several difficulties, all ultimately due to the fact that  $H_*(Z_\lambda)$  is different from both  $H_*(Z_0)$  and  $H_*(X) \oplus H_*(Y)$ . This is sorted out in Section 10, where we define the convolution operation and prove a first Symplectic Sum Theorem: formula (0.1) holds when all stable maps are  $\delta$ -flat.

In Sections 11 and 12 we remove the flatness assumption by partitioning the neck into a large number of segments and using the pigeon-hole principle as in Wieczorek [W]. For that we

construct spaces  $Z_{\lambda}^{N}(\mu_{1},\ldots,\mu_{2N+1})$ , each symplectically isotopic to  $Z_{\lambda}$ . As  $(\mu_{1},\ldots,\mu_{2N+1})\to 0$ 

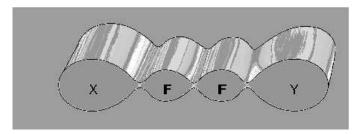


Figure 2:  $Z_{\lambda}(\mu, \mu, \mu)$  for  $|\mu| \ll |\lambda|$ 

these degenerate to the singular space obtained by connecting X to Y through a series of 2N copies of the rational ruled manifold  $\mathbb{F}_V$  obtained by adding an infinity section to the normal bundle to V. An energy bound shows that for large N each map into  $Z_{\lambda}(\mu_1, \ldots, \mu_{2N+1})$  must be flat in most necks. Squeezing some or all of the flat necks decomposes the curves in  $Z_{\lambda}$  into curves in X joined to curves in Y by a chain of curves in intermediate spaces  $\mathbb{F}_V$ . The limit maps are then  $\delta$ -flat, so formula (0.1) applies to each. This process counts each stable map many times (there are many choices of where to squeeze) and in fact gives an open cover of the moduli space. Working through the combinatorics and inverting a power series, we show that the total contribution of the entire neck region between X and Y is given by a certain TW invariant of  $\mathbb{F}_V$  — the S-matrix of Definition (11.3).

The S-matrix keeps track of how the genus, homology class, and intersection points with V change as the images of stable maps pass through the middle region of Figure 2. Observing this back in the model of Figure 1, one sees these quantities changing abruptly as the map passes through the neck — the maps are "scattered" by the neck. The scattering occurs when some of the stable maps contributing to the TW invariant of  $Z_{\lambda}$  have components that lie entirely in V in the limit as  $\lambda \to 0$ . Those maps are not V-regular, so are not counted in the relative invariants of X or Y. But by moving to the spaces of Figure 2 this complication can be analysized and related to the relative invariants of the ruled manifold  $\mathbb{F}_{V}$ .

The S-matrix is the final subtlety. With it in hand, we can at last state our main result.

**Symplectic Sum Theorem** Let Z be the symplectic sum of (X, V) and (Y, V) and suppose that  $\alpha \in \mathbb{T}(H_*(Z))$  splits as  $(\alpha_X, \alpha_Y)$  as in Definition 10.5. Then the TW invariant of Z is given in terms of the relative invariants of X and Y by

$$TW_Z(\alpha) = TW_X^V(\alpha_X) * S_V * TW_Y^V(\alpha_Y)$$
(0.2)

where \* is the convolution operation (10.6) and  $S_V$  is the S-matrix (11.3).

A detailed statement of this theorem is given in Section 12 and its extension to general constraints  $\alpha$  is discussed in Section 13. We actually state and prove (0.2) as a formula for the *relative* invariants of Z in terms of the relative invariants of X and Y (Theorem 12.3). In that form the formula can be iterated.

Of course, (0.2) is of limited use unless we can compute the relative invariants of X and Y and the associated S-matrix. That turns out to be perfectly feasible, at least for simple spaces. In Section 14 we build a collection of two and four dimensional spaces whose relative GW invariants

we can compute. We also prove that the S-matrix is the identity in several cases of particular interest.

The last section presents applications. The examples of section 14 are used as building blocks to give short proofs of three recent results in enumerative geometry: (a) the Caporaso-Harris formula for the number of nodal curves in  $\mathbb{P}^2$  [CH], (b) the formula for the Hurwitz numbers counting branched covers of  $\mathbb{P}^1$  ([GJV] [LZZ]), and (c) the "quasimodular form" expression for the rational enumerative invariants of the rational elliptic surface ([BL]). In hindsight, our proofs of (a) and (b) are essentially the same as those in the literature; using the symplectic sum formula makes the proof considerably shorter and more transparent, but the key ideas are the same. Our proof of (c), however, is completely different from that of Bryan and Leung in [BL]. It is worth outlining here.

The rational elliptic surface E fibers over  $\mathbb{P}^1$  with a section s and fiber f. For each  $d \geq 0$  consider the invariant  $GW_d$  which counts the number of connected rational stable maps in the class s + df. Bryan and Leung showed that the generating series  $F_0(t) = \sum GW_d t^d$  is

$$F_0(t) = \left(\prod_d \frac{1}{1 - t^d}\right)^{12}. \tag{0.3}$$

This formula is related to the work of Yau-Zaslow [YZ] and is one of the simplest instances of some general conjectures concerning counts of nodal curves in complex surfaces — see [Go].

While the intriguing form (0.3) appears in ([BL]) for purely combinatorial reasons, it arises in our proof because of a connection with elliptic curves. In fact, our proof begins by relating  $F_0$  to a similar series H which counts elliptic curves in E. We then regard E as the fiber sum  $E\#(T^2\times S^2)$  and apply the symplectic sum formula. The relevant relative invariant on the  $T^2\times S^2$  side is easily seen to the generating function G(t) for the number of degree d coverings of the torus  $T^2$  by the torus. The symplectic sum formula reduces to a differential equation relating  $F_0(t)$  with G(t), and integration yields the quasimodular form (0.3). The details, given in section 15.3, are rather formal; the needed geometric input is mostly contained in the symplectic sum formula.

All three of the applications in section 15 use the idea of 'splitting the target' mentioned at the beginning of this introduction. Moreover, all three follow from rather simple cases of the Symplectic Sum Theorem — cases where the S-matrix is the identity and where at least one of the relative invariants in (0.2) is readily computed using elementary methods. The full strength of the symplectic sum theorem has not yet been used.

This paper is a sequel to [IP4]; together with [IP4] it gives a complete detailed exposition of the results announced in [IP3]. Further applications have already appeared in [IP2] and [I]. Li and Ruan also have a sum formula [LR]. Eliashberg, Givental, and Hofer are developing a general theory for invariants of symplectic manifolds glued along contact boundaries [EGH].

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Appendix: Expansions of Relative TW Invariants

#### 1 GW and TW Invariants

For stable maps and their associated invariants we will use the definitions and notation of [IP4]; those are based on the Gromov-Witten invariants as defined by Ruan-Tian [RT1] and Li-Tian [LT]. In summary, the definition goes as follows. A bubble domain B is a finite connected union of smooth oriented 2-manifolds  $B_i$  joined at nodes together with n marked points, none of which are nodes. Collapsing the unstable components to points gives a connected domain st(B). Let  $\overline{\mathcal{U}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$  be the universal curve over the Deligne-Mumford space of genus g curves with g marked points. We can put a complex structure g on g by specifying an orientation-preserving map g is g and use the notation g in g instead of g. We will often write g for the curve g and use the notation g instead of g.

A  $(J, \nu)$ -holomorphic map from B is then a map  $(f, \phi): B \to X \times \overline{\mathcal{U}}_{g,n}$  where  $\phi = \phi_0 \circ st$  and which satisfies  $\bar{\partial}_J f = \phi^* \nu$  on each component  $B_i$  of B. A stable map is a  $(J, \nu)$ -holomorphic map for which the energy

$$E(f,\phi) = \frac{1}{2} \int |d\phi|^2 + |df|^2 \tag{1.1}$$

is positive on each component  $B^i$ . This means that each component  $B_i$  is either a stable curve or the restriction of f to  $B_i$  is non-trivial in homology.

For generic  $(J, \nu)$  the moduli space  $\mathcal{M}_{g,n}(X, A)$  of stable  $(J, \nu)$ -holomorphic maps representing a class  $A \in H_2(X)$  is a smooth orbifold of (real) dimension

$$-2K_X[A] - \frac{1}{2}(\dim X - 6)\chi + 2n \tag{1.2}$$

Its compactification carries a (virtual) fundamental class whose pushforward under the map

$$\overline{\mathcal{M}}_{g,n}(X,A) \stackrel{\operatorname{st}\times\operatorname{ev}}{\longrightarrow} \overline{\mathcal{M}}_{g,n}\times X^n$$

defined by stabilization and evaluation at the marked points is the Gromov-Witten invariant  $GW_{X,A,g,n} \in H_*(\overline{\mathcal{M}}_{g,n} \times X^n)$ . These can be assembled into a single invariant by setting  $\overline{\mathcal{M}} = \bigcup_{g,n} \overline{\mathcal{M}}_{g,n}$ , and introducing variables  $\lambda$  to keep track of the euler class and  $t_A$  satisfying  $t_A t_B = t_{A+B}$  to keep track of A. The total GW invariant of  $(X,\omega)$  is then the formal series

$$GW_X = \sum_{A,g,n} \frac{1}{n!} GW_{X,A,g,n} \ t_A \ \lambda^{2g-2}. \tag{1.3}$$

whose coefficients lie in  $H^*(\overline{\mathcal{M}}) \otimes \mathbb{T}(X)$  where  $\mathbb{T}(X)$  denotes the total tensor algebra  $\mathbb{T}(H^*(X))$ . This in turn defines the "Taubes-Witten" invariant

$$TW_X = e^{GW_X}$$

whose coefficients count holomorphic curves whose domains need not be connected (as occur in [T]).

The dimension (1.2) is the index of the linearization the  $(J, \nu)$ -holomorphic equation, which is obtained as follows. A variation of a map f is specified by a  $\xi \in \Gamma(f^*TX)$ , thought of as a vector field along the image, and a variation in the complex structure of  $C = (B, j, x_1, \ldots, x_n)$  is specified by

$$k \in T_C \mathcal{M}_{g,n} \cong H_j^{0,1} \left( TB \otimes \mathcal{O} \left( -\sum x_i \right) \right)$$
 (1.4)

(tensoring with  $\mathcal{O}(-x)$  accounts for the variation in the marked point x). Calculating the variation in the path

$$(f_t, j_t) = \left(\exp_f(t\xi), \ j + tk\right) \tag{1.5}$$

one finds that the linearization at (f, j) is the operator

$$D_{f,j}: \Gamma(f^*TX) \oplus T_C \mathcal{M}_{q,n} \to \Lambda^{0,1}(f^*TX)$$
(1.6)

given by  $D_{f,j}(\xi,k) = L(\xi) + Jf_*k$  with

$$L(\xi)(w) = \frac{1}{2} \left[ \nabla_w \xi + J \nabla_{jw} \xi + \frac{1}{2} (\nabla_\xi J) (f_* j w + J f_* w - 2J \nu(w)) \right] - (\nabla_\xi \nu)(w)$$
 (1.7)

where w is a vector tangent to the domain and  $\nabla$  is the pullback connection on  $f^*TX$ . Writing L as the sum of its J-linear component  $\frac{1}{2}(L+JLJ)=\overline{\partial}_f+S$  and its J-antilinear component T, we have

$$L(\xi)(w) = \overline{\partial}_{f,j}\xi(w) + S(\xi, f_*w, f_*jw, w) + T(\xi, f_*w, f_*jw, w).$$
(1.8)

Here  $\overline{\partial} = \sigma_J \circ \nabla$  with  $\sigma_J$  the *J*-linear part of the symbol of *L*, *T* is the tensor on  $X \times \mathcal{U}$  with  $JT(\xi, X, Y, w)$  given by

$$\frac{1}{2}\left[\left(\nabla_XJ\right)+J(\nabla_YJ)\right]\xi+\frac{1}{4}\left[\left(\nabla_{J\xi}J\right)-J(\nabla_\xi J)\right]\left(Y+JX-2J\nu(w)\right)+\left(\nabla_{J\xi}\nu-J\nabla_\xi\nu\right)(w)$$

and S is a similarly looking tensor. Note that since the first two terms of L are complex linear we have, for complex valued functions  $\phi$ ,

$$L(\phi\xi) = \overline{\partial}\phi \cdot \xi + \phi L(\xi) + (\overline{\phi} - \phi)T(\xi). \tag{1.9}$$

The invariant  $GW_X$  was generalized in [IP4] to an invariant of  $(X, \omega)$  relative to a codimension 2 symplectic submanifold V. To define it, we fix a pair  $(J, \nu)$  which is 'V-compatible' in the sense of Definition 3.2 in [IP4], that is, so that along V the normal components of  $\nu$  and of the tensor T in (1.8) satisfy

(a) 
$$V$$
 is  $J$ -invariant and  $\nu^N = 0$ , and (1.10)

(b) 
$$T^N(\xi, X, JX - \nu, w) = 0$$
 for all  $\xi \in N_V, X \in TV$  and  $w \in TC$ .

A stable map into X is called V-regular if no component of the domain is mapped entirely into V and no marked point or node is mapped into V. Any such map has only finitely many points  $x_1, \ldots, x_\ell$  in  $f^{-1}(V)$ . After numbering these, their degrees of contact with V define a multiplicity vector  $s = (s_1, \ldots, s_\ell)$  and three associated integers:

$$\ell(s) = \ell, \qquad \deg s = \sum s_i, \qquad |s| = \prod s_i.$$
 (1.11)

The space of all V-regular maps is the union of components

$$\mathcal{M}^{V}_{\chi,n,s}(X,A) \subset \mathcal{M}_{\chi,n+\ell}(X,A)$$

labeled by vectors s of length  $\ell(s)$ . This has a compactification that comes with 'evaluation' maps

$$\varepsilon_V : \overline{\mathcal{M}}_{\chi,n,s}^V(X,A) \to \widetilde{\mathcal{M}}_{\chi,n} \times X^n \times \mathcal{H}_{X,A,s}^V.$$
 (1.12)

Here  $\widetilde{\mathcal{M}}_{\chi,n}$  is the space of stable curves with finitely many components, Euler class  $\chi$  and n marked points, and  $\mathcal{H}^V_{X,A,s}$  is the 'intersection-homology' space described in section 5 of [IP4]. There is a covering map  $\varepsilon: \mathcal{H}^V_{X,A,s} \to H_2(X) \times V_s$  whose first component records the class A and whose component in the space  $V_s \cong V^{\ell(s)}$  records the image of the last  $\ell(s)$  marked points. This covering is a necessary complication to the definition of relative GW invariants.

The complication occurs because of "rim tori". A rim torus is an element of

$$\mathcal{R} = \ker \left( \iota_* : H_2(X \setminus V) \to H_2(X) \right) \tag{1.13}$$

where  $\iota$  is the inclusion. Each such element can be represented as  $\pi^{-1}(\gamma)$  where  $\pi$  is the projection  $S_V \to V$  from the boundary of a tubular neighborhood of V (the "rim of V") and  $\gamma: S^1 \to V$  is a loop in V. The group  $\mathcal{R}$  is the group of deck transformation of the covering

$$\mathcal{R} \longrightarrow \mathcal{H}_X^V \\
\downarrow \varepsilon \\
H_2(X) \times \coprod_s V_s. \tag{1.14}$$

When there are no rim tori (as is the case if V is simply connected)  $\mathcal{H}_{X,s}^V$  reduces to  $H_2(X) \times V_s$  and the evaluation map (1.12) is more easily described.

The tangent space to  $\mathcal{M}_{\chi,n,s}^V(X,A)$  is modeled on ker  $D_s$  where  $D_s$  is the restriction of (1.6) to the subspace where  $\xi^N$  has a zero of order  $s_i$  at the marked points  $x_i$ ,  $i = 1, \ldots, \ell$ . It follows that

$$\dim \mathcal{M}_{\chi,n,s}^{V}(X,A) = -2K_X[A] - \frac{\chi}{2} (\dim X - 6) + 2n - 2(\deg s - \ell(s))$$
 (1.15)

With this understood, the definition of the relative GW invariant parallels the above definition of  $GW_X$ : the image moduli space under (1.12) carries a homology class which, after summing on  $\chi$ , n and s, can be thought of as a map

$$GW_{X,A}^{V}: \mathbb{T}(H^{*}(X)) \longrightarrow H_{*}(\overline{\mathcal{M}} \times \mathcal{H}_{X}^{V}; \mathbb{Q}[\lambda]).$$
 (1.16)

This gives the expansion

$$GW_X^V = \sum_{A, g} \sum_{\substack{s \text{ ordered seq} \\ \deg s = A \cdot V}} \frac{1}{\ell(s)!} GW_{X,A,g,s}^V t_A \lambda^{2g-2}$$

$$\tag{1.17}$$

whose coefficients are (multi)-linear maps  $\mathbb{T}(H^*(X)) \to H_*(\overline{\mathcal{M}} \times \mathcal{H}^V_{X,A,s})$  (dividing by  $\ell(s)$ ! eliminates the redundancy associated with renumbering the last  $\ell$  marked points). The corresponding relative Taubes-Witten invariant is again given by

$$TW_X^V = \exp(GW_X^V). \tag{1.18}$$

After imposing constraints one can expand  $TW_X^V$  in power series. That is done in the appendix under the assumption that there are no rim tori.

### 2 Symplectic Sums

Assume X and Y are 2n-dimensional symplectic manifolds each containing symplectomorphic copies of a codimension two symplectic submanifold  $(V, \omega_V)$ . Then the normal bundles are oriented, and we assume they have opposite Euler classes:

$$e(N_X V) + e(N_Y V) = 0.$$
 (2.1)

We can then fix a symplectic bundle isomorphism  $\psi: N_X^*V \to N_YV$ .

This data determines a family of symplectic sums  $Z_{\lambda} = X \#_{V,\lambda} Y$  parameterized by  $\lambda$  near 0 in  $\mathbb{C}$ ; these have been described in [Gf] and [MW]. In fact, this family fits together to form a smooth 2n+2-dimensional symplectic manifold Z that fibers over a disk. In this section we will construct Z and describe its properties.

**Theorem 2.1** Given the above data, there exists a 2n+2-dimensional symplectic manifold  $(Z,\omega)$  and a fibration  $\lambda: Z \to D$  over a disk  $D \subset \mathbb{C}$ . The center fiber  $Z_0$  is the singular symplectic manifold  $X \cup_V Y$ , while for  $\lambda \neq 0$ , the fibers  $Z_{\lambda}$  are smooth compact symplectic submanifolds—the symplectic connect sums.

This displays the  $Z_{\lambda}$  as deformations, in the symplectic category, of the singular space  $X \cup_V Y$ . For  $\lambda \neq 0$  these are symplectically isotopic to one another and to the sums described in [Gf] and [MW].

The proof of Theorem 2.1 involves the following construction. Given a complex line bundle  $\pi:L\to V$  over V, fix a hermitian metric on L, set  $\rho(x)=\frac{1}{2}|x|^2$  for  $v\in L$ , and choose a compatible connection on L. The connection defines a real-valued 1-form  $\alpha$  on  $L\setminus \{\text{zero section}\}$  with  $\alpha(\partial/\partial\theta)=1$  (identify the principal bundle with the unit circle bundle and pull back the connection form by the radial projection). The curvature F of  $\alpha$  pulls back to  $\pi^*F=d\alpha$ . Then the 2-form

$$\omega = \pi^*(\omega_V) + \rho \pi^*(F) + d\rho \wedge \alpha \tag{2.2}$$

is  $S^1$ -invariant, closed, and non-degenerate for small  $\rho$ . The moment map for the circle action  $v\mapsto e^{i\theta}$  is the function  $-\rho$  because  $i_{\frac{\partial}{\partial \theta}}\omega = i_{\frac{\partial}{\partial \theta}}(d\rho\wedge\alpha) = -d\rho$ .

We can extend  $\omega$  to a compatible triple  $(\omega, J, g)$  as follows. Fix a metric  $g^V$  and an almost complex structure  $J_V$  on V compatible with  $\omega_V$  in the sense that

$$g_V(X,Y) = \omega_V(X,J_VY)$$

for all tangent vectors X and Y. At each  $x \in L \setminus \{\text{zero section}\}$ , there is a splitting  $T_x L = V \oplus H$  into a vertical subspace  $V = \ker \pi_*$  and a horizontal subspace  $H = \ker d\rho \cap \ker \alpha$ . We can therefore identify  $V = L_x$  and  $H = T_{\pi(x)}V$  and define an almost complex structure on the total space of L by  $J = J_L \oplus J_V$ . Writing r(x) = |x| and  $F_J(X,Y) = F(X,JY)$ , one can then check that the metric

$$g = \pi^*(g^V + \rho F_J) + (dr)^2 + r^2 \alpha \otimes \alpha \tag{2.3}$$

is compatible with J and  $\omega$ .

The dual bundle  $L^*$  has a dual metric  $\rho^*(v^*) = \frac{1}{2}|v^*|^2$  and connection  $\alpha^*$  with curvature -F. This gives a symplectic form similar to (2.2) on  $L^*$  and hence one on  $\pi: L \oplus L^* \to V$ , namely

$$\omega = \pi^* [\omega_V + (\rho - \rho^*) F] + d\rho \wedge \alpha - d\rho^* \wedge \alpha^*. \tag{2.4}$$

Below, we will denote points in  $L \oplus L^*$  by triples (v, x, y) where  $v \in V$  and (v, x, y) is a point in the fiber of  $L \oplus L^*$  at v. This space has

(a) a circle action 
$$(x, y) \mapsto (e^{i\theta}x, e^{-i\theta}y)$$
 with Hamiltonian  $t(v, x, y) = \rho^* - \rho$  (2.5)

(b) a natural 
$$S^1$$
 invariant map  $L \oplus L^* \to \mathbb{C}$  by  $\lambda(z, x, y) = xy \in \mathbb{C}$ .

Repeating the above construction of J and g gives an  $S^1$  invariant compatible structure  $(\omega, J, g)$  on  $L \oplus L^*$ .

**Proof of Theorem 2.1.** Let L by the complex line bundle with the same Euler class as  $N_XV$  and give L the above structure  $(\omega, J, g)$ . Using  $\psi$  and the Symplectic Neighborhood Theorem, we symplectically identify a neighborhood of V in X with the disk bundle of radius  $\varepsilon$  in L and a neighborhood of V in Y with the  $\varepsilon$ -disk bundle in  $L^*$ . Let D denote the disk of radius  $\varepsilon$  in  $\mathbb{C}$ .

The space Z is constructed from three open pieces: an X end  $E_X = (X \setminus V) \times D$ , a Y end  $E_Y = (Y \setminus V) \times D$ , and a "neck" modeled on the open set

$$N = \{ (v, x, y) \in L \oplus L^* \mid |x| \le \varepsilon, |y| \le \varepsilon \}$$

$$(2.6)$$

These are glued together by the diffeomorphisms

$$\psi_X : N \to (N_X V \setminus V) \times D$$
 by  $(v, x, y) \mapsto (v, x, \lambda(x, y))$   
 $\psi_Y : N \to (N_Y V \setminus V) \times D$  by  $(v, x, y) \mapsto (v, y, \lambda(x, y))$ 

This defines Z as a smooth manifold. The function  $\lambda$  extends over the ends as the coordinate on the D factor, giving a projection  $\lambda: Z \to D$  whose fibers are smooth submanifolds  $Z_{\lambda}$  for small  $\lambda \neq 0$ .

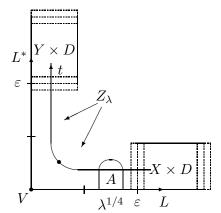


Figure 3: Construction of  $Z_{\lambda}$ 

In the region on the X side near  $|x| = \lambda^{1/4}$  (region A in Figure 3), we can merge the form  $\psi_X^*\omega$  into the symplectic form (2.4) on N by replacing  $\alpha$  by  $\eta\alpha + (1-\eta)d\theta$  where  $\eta(t)$  is a cutoff function with  $\eta = 1$  for  $|x| \leq \lambda^{1/4}$  and  $\eta = 0$  for  $|x| \geq 2\lambda^{1/4}$ . The form (2.4) then extends over the X end of Z. Doing the same on the Y side, we obtain a well-defined global symplectic form  $\omega$  on Z. The restriction of  $\omega$  to a level set  $Z_\lambda \cap U_X$  is the original symplectic form  $\omega_X$  on X; similarly, its restriction to  $Z_\lambda \cap U_Y$  is  $\omega_Y$ . Finally, along  $Z_\lambda \cap U$  we have  $\alpha^* = -\alpha$ , so  $\omega$  restricts to

$$\omega_{\lambda} = \pi^*(\omega_V - tF) - dt \wedge \alpha.$$

This is non-degenerate for small  $\lambda$ . Thus after possibly making  $\varepsilon$  smaller, we have a fibration  $\lambda: Z \to D$  with symplectic fibers.  $\square$ 

This construction shows that the neck region U of Z has a symplectic  $S^1$  action with Hamiltonian t. This action preserves  $\lambda$ , so restricts to a Hamiltonian action on each  $Z_{\lambda}$ . In fact, t gives a parameter along the neck, splitting each  $Z_{\lambda}$  into manifolds with boundary

$$Z_{\lambda} = Z_{\lambda}^{-} \cup Z_{\lambda}^{+}$$

where  $Z_{\lambda}^-$  is  $Z_{\lambda} \cup U_X$  together with the part of  $Z_{\lambda} \cup U$  with  $t \leq 0$ . From this decomposition we can recover the symplectic manifolds X and Y in two ways:

- 1. as  $\lambda \to 0, \, Z_{\lambda}^-$  (resp.  $Z_{\lambda}^+$ ) converges to X (resp. Y) as symplectic manifolds, or
- 2. X (resp. Y) is the symplectic cut of  $Z_{\lambda}^{-}$  (resp.  $Z_{\lambda}^{+}$ ) at t=0 (cf. [L]).

Thus we have collapsing maps

$$\begin{array}{ccc}
X \sqcup Y & Z_{\lambda} \\
\pi_{0} \searrow & \swarrow \pi_{\lambda} \\
& Z_{0}
\end{array} (2.7)$$

and  $\pi_{\lambda}$  is a deformation equivalence on the set where  $t \neq 0$ .

The proof of Theorem 2.1 constructs a structure  $(\omega, J, g)$  on Z whose restriction to  $Z_{\lambda}$  on the X end agrees with the given structure  $(\omega_X, J_X, g_X)$  on X (and similarly on the Y end). More generally, given V-compatible pairs  $(J_X, \nu_X)$  and  $(J_Y, \nu_Y)$  which agree along V under the the map  $\psi$  of (2.1) and with the normal components of  $\nu_X$  and  $\nu_Y$  vanishing along V, then we can extend them to  $(J, \nu)$  on the entire fibration Z.

We finish this section will a useful lemma comparing the canonical class of the symplectic sum with the canonical classes  $K_X$  and  $K_Y$  of X and Y.

**Lemma 2.2** If  $A \in H_2(Z_\lambda; \mathbb{Z})$ ,  $\lambda \neq 0$ , is homologous in Z to the union  $C_1 \cup C_2 \subset X \cup_V Y$  of cycles  $C_1$  in X and  $C_2$  in Y, then

$$K_{Z_{\lambda}}[A] = K_{Z}[A] = K_{X}[C_{1}] + K_{Y}[C_{2}] + 2\beta$$

where  $\beta$  is the intersection number  $V \cdot [C_1] = V \cdot [C_2]$ . In particular,  $K_{Z_{\lambda}}[R] = 0$  for any rimmed torus R in (1.13).

**Proof.** For  $\lambda \neq 0$ , the normal bundle to  $Z_{\lambda}$  has a nowhere-vanishing section  $\partial/\partial\lambda$ . Thus the canonical bundle of  $Z_{\lambda}$  is the restriction of the canonical bundle of Z, giving

$$K_{Z_{\lambda}}[A] = K_{Z}[A] = K_{Z}[C_{1}] + K_{Z}[C_{2}].$$

Outside the neck region of X, the tangent bundle to Z decomposes as  $TX \oplus \mathbb{C}$ . Inside the neck region we have

$$TZ = TX \oplus \pi^* \psi^* N_Y V \cong TX \oplus \pi^* (N_X V)^{-1}$$

where  $\pi$  is the projection  $N_XV \to V$ . But the Poincaré dual of V in X, regarded as an element of  $H^2_{cpt}(X)$ , is the chern class  $c_1(\pi^*N_XV)$ . Since the canonical class is minus the first chern class of the tangent bundle we conclude that

$$K_Z[C_1] = K_X[C_1] + V \cdot [C_1]$$

and similarly on the Y side.  $\square$ 

## 3 Degenerations of symplectic sums

The Gromov-Witten invariants of the symplectic sum  $Z_{\lambda}$  are defined in terms of stable pseudo-holomorphic maps from complex curves into the  $Z_{\lambda}$ . The basic idea of our connect sum formula is to approximate the maps in  $Z_{\lambda}$  by certain maps into the singular space  $Z_0$ . The first step is to understand exactly which maps into  $Z_0$  are limits of stable maps into the  $Z_{\lambda}$  as  $\lambda \to 0$ . This section gives a description of the limits of flat stable maps. This 'flat' condition, defined below, ensures that the limit has no components mapped into V.

Fix a small  $\delta > 0$ . Given a map f into  $Z_{\lambda}$ , we can restrict attention to that part of the image that lies in the ' $\delta$ -neck'

$$Z_{\lambda}(\delta) = \{ z = (v, x, y) \in Z_{\lambda} \mid ||x|^{2} - |y|^{2}| \le \delta \}.$$
(3.1)

This is a narrow region symmetric about the middle of the neck in Figure 3. The energy of f in this region is

$$E_{\delta}(f) = \frac{1}{2} \int |d\phi|^2 + |df|^2 \tag{3.2}$$

where the integral is over  $f^{-1}(Z_{\lambda}(\delta))$ .

By Lemma 1.5 of [IP4] there is a constant  $\alpha_V < 1$ , depending only on  $(J_V, \nu_V)$  such that every component of every stable  $(J_V, \nu_V)$ -holomorphic map f into V has energy

$$E(f) \ge \alpha_V. \tag{3.3}$$

**Definition 3.1 (Flat Maps)** A stable  $(J, \nu)$ -holomorphic map f into Z is flat (more precisely  $\delta$ -flat) if the energy in the  $\delta$ -neck is at most half  $\alpha_V$ , that is

$$E_{\delta}(f) \leq \alpha_V/2. \tag{3.4}$$

For each small  $\lambda$ , let

$$\mathcal{M}_{\gamma,n}^{flat}(Z_{\lambda},A)$$

denote the set of flat maps in  $\mathcal{M}_{\chi,n}(Z_{\lambda},A)$ . These are a family of subsets of the space of stable maps and we write

$$\lim_{\lambda \to 0} \mathcal{M}_{\chi,n}^{flat}(Z_{\lambda}, A) \tag{3.5}$$

for the set of limits of sequences of flat maps into  $Z_{\lambda}$  as  $\lambda \to 0$ . Because (3.4) is a closed condition this limit set is a *closed* subspace of  $\overline{\mathcal{M}}(Z)$ . The remainder of this section is devoted to a precise description of the space (3.5).

**Lemma 3.2** Each element of (3.5) is a stable map f to  $Z_0 = X \cup_V Y$  with no irreducible components of the domain mapped entirely into V.

**Proof.** Each sequence in (3.5) has a subsequence  $f_k$  converging in the space of stable maps  $\overline{\mathcal{M}}_{\chi,n}(Z,A)$  to a limit  $f:C\to Z$ . In particular, the images converge pointwise, so lie in  $Z_0$ .

Suppose that the image of some component  $C_i$  of C lies in V. Then the restriction  $f_i$  of f to that component satisfies  $E(f_i) \leq E_{\delta}(f)$ . Furthermore, by Theorem 1.6 of [IP4] the sequence  $f_k$  (after precomposing with diffeomorphisms) converges in  $C^0$  and in  $L^{1,2}$ , so  $E_{\delta}(f) = \lim E_{\delta}(f_k) \leq \alpha_V/2$ . This contradicts (3.3).  $\square$ 

We can be very specific about how the images of the maps in (3.5) hit V. By Lemma 3.2 and Lemma 3.4 of [IP4], at each point  $p \in f^{-1}(V)$  the normal component of f has a local expansion  $a_0 z^d + \ldots$  This defines a local 'degree of contact' with V

$$d = \deg(f, p) \ge 1 \tag{3.6}$$

and implies that  $f^{-1}(V)$  is a finite set of points. Restricting f to one component  $C_i$  of C and removing the points  $f^{-1}(V)$  gives a map from a connected domain to the disjoint union of  $X \setminus V$  and  $Y \setminus V$ . Thus the components of C are of two types: those components  $C_i^X$  whose image lies in X, and those components  $C_i^Y$  whose image lies in Y. We can therefore split f into two parts: the union of the components whose image lies in X defines a map  $f_1: C_1 \to X$ , from a (possibly disconnected, prestable) curve  $C_1$ , and the remaining components define a similar map  $f_2: C_2 \to Y$ .

**Lemma 3.3**  $f^{-1}(V)$  consists of nodes of C. For each node  $x = y \in f^{-1}(V)$ 

$$\deg(f_1, x) = \deg(f_2, y).$$

**Proof.** The local degree (3.6) is a linking number. Specifically, let  $N_X(V)$  be a tubular neighborhood of V in X and let  $\mu_X$  be the generator of  $H_1(N_X(V) \setminus V) = \mathbb{Z}$  oriented as the boundary of a holomorphic disk normal to V. If  $\mu_Y$  is the corresponding generator on the Y side, then  $\mu_X = -\mu_Y$  in  $H_1$  of the neck  $Z_\lambda(\delta)$ . For each point x in  $f_1^{-1}(V)$  and each small circle  $S_\varepsilon$  around x, the local degree d satisfies

$$d \cdot \mu = [f_1(S_{\varepsilon})].$$

If x is not a node of C then by Theorem 1.6 of [IP4]  $f_k$  converges to  $f_1$  in  $C^1$  in a disk D around x. But then for large k  $d \cdot \mu = [f(S_{\varepsilon})] = [f_k(S_{\varepsilon})] = [f_k(\partial D)] = 0$ , contradicting (3.6).

Next consider a node x=y of C which is mapped into V. Choose holomorphic disks  $D_1=D(x,\varepsilon)$  and  $D_2=D(y,\varepsilon)$  that contain no other points of  $f^{-1}(V)$  and let  $S_i=\partial D_i$ . Then  $S_1\cup S_2$  bounds in C, so  $[f_k(S_1)]+[f_k(S_2)]=0$  in  $H_1$  of the neck  $Z_\lambda(\delta)$ . Again,  $f_k\to f$  in  $C^0$ , so  $0=[f(S_1)]+[f(S_2)]=d_1\mu_1+d_2\mu_2$  where  $\mu_i$  is either  $\mu_X$  or  $\mu_Y$ , depending on which side  $f(S_i)$  lies. Since  $d_i>0$  the only possibility is that x=y is a node between a component in X and one in Y and  $d_1=d_2$ .  $\square$ 

Lemmas 3.2 and 3.3 show that each map f in the limiting set (3.5) splits into  $(J, \nu)$ holomorphic maps  $f_1: C_1 \to X$  and  $f_2: C_2 \to Y$ . Numbering the nodes in  $f^{-1}(V)$  gives
extra marked points  $x_1, \ldots, x_\ell$  on  $C_1$  and matched  $y_1, \ldots, y_\ell$  on  $C_2$  with  $s_i = \deg x_i = \deg y_i$ .
Furthermore, the Euler characteristics  $\chi_1$  of  $C_1$  and  $\chi_2$  of  $C_2$  satisfy

$$\chi_1 + \chi_2 - 2\ell = \chi. \tag{3.7}$$

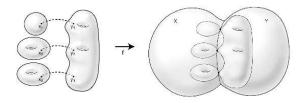


Figure 4: The map  $f_0 = (f_1, f_2)$  into  $Z_0 = X \cup_V Y$ 

Remark 3.4 To simplify the exposition we will assume for the rest of the paper that (a) all the components are stable, and (b)  $C_1$  and  $C_2$  have no non-trivial automorphisms. In fact, one can deal with unstable components by first stabilizing as in [LT], and deal with automorphisms by lifting to a cover of the universal curve as in [RT2], section 2. One can easily check that the analytic arguments below, which are local on the moduli space of holomorphic maps, carry through the stabilization and lifting procedures. Under these assumptions the moduli spaces are generically smooth and their virtual fundamental class is equal to the actual fundamental class after taking quotients as in [RT2].

We can now give a global description of how the limit maps f in (3.5) are assembled from their components  $f_1$  and  $f_2$ . First, consider how the domain curves fit together in accordance with (3.7). Given stable curves  $C_1$  and  $C_2$  (not necessarily connected) with Euler characteristics  $\chi_i$  and  $n_i + \ell$  marked points, we can construct a new curve by identifying the last  $\ell$  marked points

of  $C_1$  with the last  $\ell$  marked points of  $C_2$ , and then forgetting the marking of these new nodes. This defines an attaching map

$$\xi_{\ell}: \widetilde{\mathcal{M}}_{\chi_1, n_1 + \ell} \times \widetilde{\mathcal{M}}_{\chi_2, n_2 + \ell} \longrightarrow \widetilde{\mathcal{M}}_{\chi_1 + \chi_2 - 2\ell, n_1 + n_2} \tag{3.8}$$

whose image is a subvariety of complex codimension  $\ell$ . Taking the union over all  $\chi_1, \chi_2, n_1$  and  $n_2$  gives an attaching map  $\xi_{\ell} : \widetilde{\mathcal{M}} \times \widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}$  for each  $\ell$ .

Second, consider how the maps fit together along V. The evaluation map

$$\operatorname{ev}_s: \mathcal{M}^V_{Y,n,s}(X) \times \mathcal{M}^V_{Y,n,s}(Y) \stackrel{\varepsilon_V \times \varepsilon_V}{\longrightarrow} \mathcal{H}^V_X \times \mathcal{H}^V_Y \stackrel{\varepsilon_2 \times \varepsilon_2}{\longrightarrow} V_s \times V_s.$$

records the intersection points with V and the pair  $(f_1, f_2)$  lies in the space

$$\mathcal{M}^{V}(X) \underset{ev_{s}}{\times} \mathcal{M}^{V}(Y) \stackrel{\text{def}}{=} ev_{s}^{-1}(\Delta_{s}).$$
 (3.9)

where  $\Delta_s$  is the diagonal

$$\Delta_s \subset V_s \times V_s$$
.

Denote by  $\mathcal{H}_X^V \times_{\varepsilon} \mathcal{H}_Y^V = (\varepsilon_2 \times \varepsilon_2)^{-1}(\Delta)$  the fiber sum of  $\mathcal{H}_X^V$  and  $\mathcal{H}_Y^V$  along the evaluation map  $\varepsilon_2$ , where  $\Delta = \bigcup_{\varepsilon} \Delta_s$ . Then we have a well defined map

$$g: \mathcal{H}_X^V \times_{\varepsilon} \mathcal{H}_Y^V \to H_2(Z)$$
 (3.10)

which describes how the homology-intersection data of  $f_1$  and  $f_2$  determine the homology class of f.

**Lemma 3.5** For generic  $(J, \nu)$  the space (3.9) is a smooth orbifold of the same dimension as  $\mathcal{M}_{\chi,n}^{flat}(Z_{\lambda}, A)$  given by (1.2).

**Proof.** The dimensions of  $\mathcal{M}_{\chi_1,s}^V(X,A_1)$  and  $\mathcal{M}_{\chi_2,s}^V(Y,A_2)$  are given by (1.15). A small modification of the proof of Lemma 8.6 of [IP4] shows that the evaluation map at the last  $\ell = \ell(s)$  marked points (i.e. the intersection points with V) is transversal to the diagonal  $\Delta \subset V^{\ell} \times V^{\ell}$ , imposing  $\ell \dim V = \ell(\dim X - 2)$  conditions. Thus (3.9) is a smooth manifold of dimension

$$-2K_X[A_1] - 2K_Y[A_2] - 4\deg s - \frac{1}{2}(\dim X - 6)(\chi_1 + \chi_2 - 2\ell) + 2n.$$

The lemma follows by comparing this with (1.2) using (3.7), Lemma 2.2, and the fact that  $\deg s = A_1 \cdot V = A_2 \cdot V$ .  $\square$ 

Finally, note that renumbering the pairs  $(x_i, y_i)$  of marked points defines a free action of the symmetric group  $S_{\ell}$  on (3.9) and the limit maps in (3.5) correspond to elements in the quotient. Moreover, after ordering the double points along V the limit set (3.5) is a closed subset

$$\mathcal{K}_{\delta} \subset \mathcal{M}^{V}(X) \underset{ev}{\times} \mathcal{M}^{V}(Y)$$
 (3.11)

which is the disjoint union of open sets labeled by s and which has compact closure as in [IP4]. Since the maps in  $\mathcal{M}^{flat}(Z_{\lambda})$  are  $C^0$  close to flat maps into  $Z_0$  for small  $\lambda$  there is a decomposition

$$\mathcal{M}^{flat}(Z_{\lambda}) \; = \; \bigsqcup_{s} \left( \mathcal{M}^{flat}_{s}(Z_{\lambda}) \right) \Big/ \, S_{\ell(s)}$$

as a union of components labeled by ordered sequences  $s = (s_1, s_2...)$ . As in the proof of Lemma 3.3, these  $s_i$  are local winding numbers of the  $\ell(s)$  vanishing cycles  $S_{\varepsilon}$ . In that form the labeling extends to all continuous maps  $C^0$  close to flat maps into  $Z_0$ . Thus for small  $\lambda$ 

$$\mathcal{M}_s^{flat}(Z_\lambda) \subset \operatorname{Map}_s(Z_\lambda)$$

where  $\operatorname{Map}_s(Z_{\lambda})$ , the "space of labeled maps", is the set of labeled continuous maps into  $Z_{\lambda}$  which are  $C^0$  close to flat maps into  $Z_0$ .

Thus with this notation, the statements of Lemmas 3.2 and 3.3 translate into the commutative diagram

$$\bigsqcup_{s} \mathcal{M}^{V}(X) \underset{evs}{\times} \mathcal{M}^{V}(Y) \qquad \longleftarrow \qquad \lim_{\lambda} \left( \bigsqcup_{s} \mathcal{M}_{s}^{flat}(Z_{\lambda}) \right) \\
\downarrow \qquad \qquad \qquad \qquad \downarrow \\
(\widetilde{\mathcal{M}} \times \widetilde{\mathcal{M}}) \times \left( \mathcal{H}_{X} \underset{\varepsilon}{\times} \mathcal{H}_{Y} \right) \stackrel{\xi_{\ell(s)} \times g}{\longrightarrow} \qquad \widetilde{\mathcal{M}} \times H_{2}(Z). \tag{3.12}$$

The top arrow shows how the maps that arise as limits of flat maps decompose into pairs  $(f_1, f_2)$  of V-regular maps into X and Y, while the bottom arrow keeps track of the domains and homology classes (the vertical maps arise from (1.12) in the obvious way).

One then expects the top arrow in (3.12) to be a diffeomorphism for each s and both sides to be a model for the stable maps into  $Z_{\lambda}$  for that s. The analysis of the next six sections will show that this is true after passing to a finite cover.

The necessity of passing to covers is dictated by clustering phenomenon mentioned in the introduction: when s>1 each curve in  $Z_0$  is close (in the stable map topology) to a cluster of curves in  $Z_{\lambda}$  for small  $\lambda$ , and these coalesce as  $\lambda \to 0$ . To distinguish the curves within a cluster and indeed to even verify this statement about clustering, it is necessary to use stronger norms and distances — strong enough that the distances between the maps within a cluster are bounded away from zero as  $\lambda \to 0$ . The maps in a cluster can then be thought of as the fiber of a covering of the space of limit maps. The next three sections introduce the required norms and construct a first version of the covering. The first step is to define an appropriate distance function on the space of stable curves.

## 4 The Space of Curves

Given two holomorphic maps, one can measure the distance between their domain curves using some metric on the Deligne-Mumford space  $\overline{\mathcal{M}}_{g,n}$ . However, it is often more convenient to fix a diffeomorphism of the domains, regarding the two curves as two complex structures j and j' on a single 2-manifold, and measuring the distance between j and j' using a Sobolev norm. In this section we will define diffeomorphisms between nearby curves in the universal family, fix a Sobolev metric, and describe the corresponding distance function on  $\overline{\mathcal{M}}_{g,n}$ .

Our distance function is designed so that a neighborhood of the image of the attaching map (3.8) is obtained by gluing cylindrical ends of the spaces  $\mathcal{M}_{g,n}$ . It is a complete metric on  $\overline{\mathcal{M}}_{g,n} \setminus \mathcal{N}$  where  $\mathcal{N}$  is the set of all nodal curves; in particular it is stronger than the Weil-Petersson metric.

The construction starts by fixing a Riemannian metric  $g_{\mathcal{U}}$  on the universal curve  $\overline{\mathcal{U}}_{g,n} \xrightarrow{\pi} \mathcal{M}_{g,n}$  compatible with the complex structure. In the fibers of  $\overline{\mathcal{U}}_{g,n}$  the 'special points' (marked points

and nodes) are distinct and hence, by compactness, are separated by a minimum distance. After conformally changing the metric we can assume that the separation distance is at least 4 and that every fiber is flat in the disk of radius 3 around each of its nodes. We also fix a smooth function  $\tilde{\rho}$  on  $\overline{\mathcal{U}}_{g,n}$  equal to the distance to the node in these disks of radius 3. Finally, we replace  $g_{\mathcal{U}}$  by a conformal metric that is singular along the nodal points locus namely

$$g = \widetilde{\rho}^{-2}g_{\mathcal{U}} \tag{4.1}$$

To understand the geometry of this metric we focus attention to a small ball U in the set  $\mathcal{N}_{\ell}$  of  $\ell$  nodal curves and construct a local model. Each  $C_0 = C_0(u) = \pi^{-1}(u)$  with  $u \in U$  is the union of not-necessarily-connected curves  $C_1$  and  $C_2$  intersecting at the nodes where points  $x_k \in C_1$  are identified with  $y_k \in C_2$  for  $k = 1, ..., \ell$ . For each k we fix local coordinates  $\{z_k\}$  on  $C_1$  and  $\{w_k\}$  on  $C_2$  centered at the nodes. We can use the construction of Section 1 to form a family of symplectic sums; for details see [Ma]. The result is a holomorphic fibration

$$\mathcal{F} \to U \times D^{\ell}$$

whose fibers  $C_{\mu}(u)$  are given by  $z_k w_k = \mu_k$  in disjoint balls  $B_k$  centered on the nodes and which has a fixed a trivialization outside a neighborhood of the nodes (here  $D^{\ell}$  is the unit disk in  $\mathbb{C}^{\ell}$ ).

**Remark 4.1** The gluing parameters  $\{\mu_k\}$  are intrinsically elements of the bundle

$$\bigoplus_{k=1}^{\ell} \left( \mathcal{L}_k \otimes \mathcal{L}_k' \right)^* \tag{4.2}$$

where  $\mathcal{L}_k$  and  $\mathcal{L}'_k$  the relative cotangent bundle to  $C_1$  at  $x_k$  and respectively to  $C_2$  at  $y_k$ . Thus (4.2) models the tubular neighborhood of  $\mathcal{N}_\ell$  in  $\overline{\mathcal{M}}_{g,n}$ .

Fix a metric on  $\mathcal{F}$  which is euclidean in the coordinates  $(z_k, w_k)$  on each  $B_k$  and  $B_k$  has radius at least 4. The induced metric on  $C_{\mu} \cap B_k$  is

$$g_{\mu} = dz \, d\bar{z} + dw \, d\bar{w} \Big|_{zw=\mu} = \left(1 + \frac{|\mu|^2}{r^4}\right) \left(dr^2 + r^2 \, d\theta^2\right)$$
 (4.3)

where r=|z| and the distance to the node in  $B_k$  is  $\rho^2=|z|^2+|w|^2=r^2+|\mu|^2/r^2$ . Switching to the conformal metric  $g=\rho^{-2}g_\mu$  as in (4.1), each nodal curve in  $\mathcal F$  has a cylindrical neck in each ball  $B_k$ . In fact, when  $\mu_k\neq 0$  we can identify  $C_\mu\cap B_k(1)$  with  $[-T,T]\times S^1$  by writing  $r=\sqrt{|\mu_k|}\ e^t$  with  $T=|\log\sqrt{|\mu_k|}|$ . In these cylindrical coordinates  $\rho^2=2|\mu|\cosh(2t)$  and

$$g = \rho^{-2} g_{\mu} = (r^{-1} dr)^2 + d\theta^2 = dt^2 + d\theta^2.$$
 (4.4)

Similarly the curves  $C_{\mu}$  have necks in  $B_k$  which become longer and longer as  $\mu_k \to 0$ .

As in [Ma] there is a biholomorphic map of fibrations from  $\mathcal{F}$  to a neighborhood of  $C_0$  in  $\overline{\mathcal{U}}_{g,n}$  (if  $C_0$  has automorphisms we lift to finite covers as in [RT2]). Because this map is holomorphic with bounded differential its restriction to each fiber is conformal and the conformal factor is bounded. Consequently, the PDE results of the next several sections, all of which involve only local considerations in the space  $\overline{\mathcal{U}}_{g,n}$ , can be done in the model space  $\mathcal{F}$  using the metric (4.4)

and the results will apply uniformly on  $\overline{\mathcal{U}}_{g,n}$ . We will henceforth consistently use this metric (4.4) on the domains of holomorphic curves. Note that the flatness condition (3.4) continues to hold (after a uniform change of constants) because the energy density is conformally invariant.

We next describe how to lift vectors in  $D^{\ell}$  to vectors on the family  $\mathcal{F} = \{C_{\mu}\} \to D^{\ell}$  around an  $\ell$ -nodal curve  $C_0$ . Fix smooth curves  $C_{\mu}, C_{\mu'} \in \mathcal{F}$ . These are identified outside a neighborhood of the nodes. That identification extends to a diffeomorphism  $\phi = \phi_{\mu\mu'} : C_{\mu} \to C_{\mu'}$  as follows. Using cylindrical coordinates  $z = \sqrt{|\mu|}e^{t+i\theta}$  around each node of  $C_{\mu}$  and  $z' = \sqrt{|\mu'|}e^{t'+i\theta'}$  around the nodes of  $C'_{\mu}$ , set

$$\phi(t,\theta) = \left(t + (2\alpha(t) - 1)(T' - T), \ \theta + \alpha(t) \arg(\frac{\mu}{\mu'})\right)$$

where  $T = \frac{1}{2}|\log |\mu||$  and where  $\alpha(t)$  is a cutoff function equal to 1 for  $t \ge 1$  and 0 for  $t \le -1$ . Note that when  $|z| \ge 1$ ,  $\phi(t,\theta) = (t+T'-T,\theta) = (t',\theta')$ , so  $\phi(z) = z'$ . The relation  $zw = \mu$  similarly implies that  $\phi(w) = w'$  whenever  $|w| \ge 1$ . Thus  $\phi$  extends as claimed.

The corresponding infinitesimal diffeomorphism defines the lifts: each  $v = (v_1, \dots, v_\ell) \in T_\mu D$  defines a family  $\mu_s = \mu + sv$  and a vector field

$$\tilde{v} = \frac{d}{ds} \phi_{\mu\mu_s}|_{s=0} = \sum_{k} \left( (\alpha - \frac{1}{2}) \operatorname{Re} \left( \frac{v_k}{\mu_k} \right), \ \alpha \cdot \operatorname{Im} \left( \frac{v_k}{\mu_k} \right) \right)$$

along  $C_{\mu}$ . Going the other way, given any path  $\mu_s$  in the complement of the nodal set  $\mathcal{N}$  we can lift the vectors  $\dot{\mu}$  as above and integrate the lifted vector fields to get diffeomorphisms  $\phi_s: C_{\mu_0} \to C_{\mu_s}$ . For each s the variation in the complex structure is  $h_s = \frac{d}{ds} \phi_s^* j$ . Define a second distance between the complex structures by

Dist 
$$(C_{\mu_0}, C_{\mu_1}) = \inf \int_0^1 ||h_s|| ds$$
 (4.5)

where the infimum is over all paths from  $\mu_0$  to  $\mu_1$  in the complement of  $\mathcal{N}$  and where

$$||h||^2 = \int_{C_{\mu}} |\nabla^2 h|^2 + |\nabla h|^2 + |h|^2. \tag{4.6}$$

Note that in each family  $\{C_{\mu}\}$  the nodal curves correspond to  $\{\mu \mid \text{some } \mu_k \text{ is zero}\}.$ 

**Lemma 4.2** On the complement of the nodal set  $\mathcal{N} = \{\mu | some \ \mu_k \ is \ zero\}$  the Riemannian metric (4.6) is uniformly equivalent to the metric

$$\sum_{k} \frac{1}{|\mu_{k}|^{2}} Re \ (d\mu_{k})^{2}. \tag{4.7}$$

**Proof.** Calculating  $h = \frac{d}{ds} (d\phi_s^{-1} \cdot j \cdot d\phi_s)$  at s = 0, one finds that

$$h = jd\tilde{v} - d\tilde{v}j = \begin{pmatrix} B & A \\ A & -B \end{pmatrix} \quad \text{where} \quad \begin{cases} A = \alpha' \operatorname{Re} \left(\frac{v}{\mu}\right) \\ B = \alpha' \operatorname{Im} \left(\frac{v}{\mu}\right). \end{cases}$$

Noting that the integrals of  $|d\alpha|$ ,  $|\nabla d\alpha|$ , and  $|\nabla^2 d\alpha|$  are independent of  $\mu$  we then have

$$||h||^2 = \sum_k \frac{2|v_k|^2}{|\mu_k|^2} \int_{-1}^1 \int_0^{2\pi} |\nabla^2 \alpha|^2 + |\nabla \alpha|^2 + |\alpha|^2 = c \sum_k \frac{|v_k|^2}{|\mu_k|^2}.$$

The metric (4.7) is cylindrical in each coordinate: using polar coordinates  $\mu=r\,e^{i\theta}$  and  $r=e^t$  we have

 $\frac{1}{|\mu|^2} \operatorname{Re} (d\mu_k)^2 = \frac{1}{r^2} \left( dr^2 + r^2 d\theta^2 \right) = dt^2 + d\theta^2.$ 

The corresponding distance function is that of the cylinder in each coordinate, so for  $\mu = r e^{i\theta} = e^{t+i\theta}$  and  $\mu' = r' e^{i\theta'} = e^{s+i\theta'}$ 

$$\operatorname{dist}^{2}(\mu, \mu') = \sum_{k} |t_{k} - t'_{k}|^{2} + |\theta_{k} - \theta'_{k}|^{2} = \sum_{k} \left| \log \left( \frac{\mu'_{k}}{\mu_{k}} \right) \right|^{2}. \tag{4.8}$$

Thus the metric (4.7), defined in a neighborhood of the nodal set  $\mathcal{N}$ , extends to a global metric on  $\mathcal{M}_{g,n} = \overline{\mathcal{M}}_{g,n} \setminus \mathcal{N}$  which is not complete —near the stratum  $\mathcal{N}_{\ell}$  of curves with  $\ell$  nodes it is asymptotic to a cylinder  $W_{\ell} \times \mathbb{R}_{+}^{\ell}$  where  $W_{\ell}$  is a bundle over  $\mathcal{N}_{\ell}$  whose fiber is the real torus  $T^{\ell}$  corresponding to the bundle (4.2). We can compactify this by identifying the end  $W_{\ell} \times \mathbb{R}_{+}^{\ell}$  with  $W_{\ell} \times (0,1)^{\ell}$  and compactifying to  $W_{\ell} \times (0,1]^{\ell}$ . This "cylindrical end compactification" projects down to the Deligne-Mumford compactification  $\overline{\mathcal{M}}_{g,n}$  so that the fiber along the nodal stratum  $\mathcal{N}_{\ell}$  is a copy of  $W_{\ell}$ .

#### 5 Renormalization at the Nodes

In this section we will consider a sequence of flat  $(J, \nu)$ -holomorphic maps

$$f_n: C_{\mu_n} \to Z_{\lambda_n} \quad \text{with} \quad \lambda_n \to 0.$$
 (5.1)

By the Compactness Theorem for holomorphic maps Z these converge to a limit map  $f_0$  from a nodal curve  $C_0$  to  $Z_0$  as described in Section 3; the convergence is in  $L^{1,2} \cap C^0$  and in  $C^{\infty}$  on compact sets in the complement of the nodes. We will refine this by constructing renormalized maps  $\hat{f}_n$  around each node and proving convergence results for the renormalized maps. This gives detailed information about how the original maps  $f_n$  are converging in a neighborhood of the nodes.

As in Section 3,  $C_0$  is the union of (not necessarily connected) curves  $C_1$  and  $C_2$  which intersect at nodes, and  $f_0$  decomposes into maps  $f_1: C_1 \to X$  and  $f_2: C_2 \to Y$ . Their images meet along V with contact vector  $s = (s_1, \ldots s_\ell)$ ; that is there are points  $x_k \in C_1$  and  $y_k \in C_2$  so that  $f_1$  and  $f_2$  contact V of order  $s_k$  at  $f_1(x_k) = f_2(y_k) \in V$ . For short, we simply write

$$f_n \rightarrow f_0 = (f_1, f_2) \in \mathcal{K} \subset \mathcal{M}_s \times_{ev} \mathcal{M}_s.$$

where  $\mathcal{K}$  is the compact set in (3.11). Note that in particular all the estimates in the next sections will be uniform on  $\mathcal{K}$ .

Around each node  $x_k$  we can use the coordinates  $(z_k, w_k)$  on the domain described before (4.3), and coordinates (v, x, y) centered on the image of the node to write  $f_n = (v_n, f_n^x, f_n^y)$  (here v is a coordinate on V and (x, y) are coordinates in the sum  $N_X \oplus N_Y$  of the normal bundles to V). These can be chosen so  $Z_\lambda$  is locally the graph of  $xy = \lambda$ . Using the expansions of  $f_0$  provided by Lemma 3.4 of [IP4] and Lemma (3.3) we can write around each node

$$f_0 = (p_k + h^v, a_k z^{s_k} + h^x, b_k w^{s_k} + h^y)$$
 where  $|h^v| \le c \rho$  and  $|h^x|, |h^y| \le c \rho^{s_k+1}$ . (5.2)

**Remark 5.1** The coefficient  $a_k$  is the  $s_k$ -jet of the function  $f^x$  at  $x_k$  modulo higher order terms, so

$$a_k \in (T_{x_k}^*C)^{s_k} \otimes N_X$$

where  $N_X$  is the pullback of the normal bundle to V in X. The evaluation map

$$\overline{\mathcal{M}}_{q,n,s}^{V} \subset \overline{\mathcal{M}}_{g,n+\ell} \to \overline{\mathcal{M}}_{g,n} \times X^{n} \times V^{\ell}$$

determines complex line bundles  $\mathcal{N}_X$  whose fiber at a map f is the normal bundle  $N_X$  to V at  $f(x_k)$ , and relative cotangent bundles  $\mathcal{L}_k$  as in (4.2) for the last  $\ell$  points of the domain. The leading coefficients are thus sections

$$a_k \in \Gamma(\mathcal{L}_k^{s_k} \otimes \mathcal{N}_X)$$
 and  $b_k \in \Gamma((\mathcal{L}_k')^{s_k} \otimes \mathcal{N}_Y).$  (5.3)

Note that the  $f_n$  are nearly holomorphic with respect to the complex structure  $J_0$  defined by the coordinates (v, x, y). In fact, from the  $(J, \nu)$ -holomorphic map equation, we have  $\overline{\partial} f_n = \overline{\partial}_{Jf_n} - (J - J_0) df_n j = \nu_n - (J - J_0) df_n j$ . Because the  $f_n$  are converging in  $C^0$  to the continuous function  $f_0$  with  $|f_0| = 0$  at t = 0, we have the pointwise bound

$$|\overline{\partial}f_n| \le c|f_n||df_n| \le \varepsilon|df_n| \tag{5.4}$$

where  $\varepsilon$  can be made arbitrarily small by restricting the domain to a small annular region in the neck of  $C_{\mu_n}$ . Similarly the metric on the target can be made arbitrarily close to the euclidean metric. In the next lemma we consider such an annular region in cylindrical coordinates  $(t, \theta)$  and estimate the energy

$$E(f_n, T) = \frac{1}{2} \int_{-T}^{T} \int_{0}^{2\pi} |df_n|^2 dt d\theta$$

**Lemma 5.2** For small r and large n, the energy E(t) = E(t, f) of  $f = f_n$  on the cylinder  $A(t) = [-t, t] \times S^1$  satisfies

$$E(t) \le E(T) \rho^{\frac{2}{3}}. \tag{5.5}$$

Consequently, there is a pointwise bound for |df| of the form

$$|df|^2 \le c_1 E(t+1) \le c_2 E \rho^{\frac{2}{3}}. \tag{5.6}$$

**Proof.** By writing f = u + iv one finds that

$$4|\overline{\partial}f|^2 dt d\theta = |df|^2 dt d\theta - 2 d(u dv).$$

Integrating over A = A(t) and using Stokes' theorem gives

$$\frac{1}{2} \int_{A} |df|^{2} = 2 \int_{A} |\overline{\partial}f|^{2} + \int_{\partial A} u \, v_{\theta} \, d\theta.$$

The boundary term is an integral over two circles. On each, we can replace u by  $\tilde{u} = u - \frac{1}{2\pi} \int_0^{2\pi} u \ d\theta$  and applying the Hölder and Poincaré inequalities on the circle

$$\int u \, v_{\theta} \, d\theta = \int \tilde{u} \, v_{\theta} \, d\theta \leq \|\tilde{u}\| \|v_{\theta}\| \leq \|\tilde{u}_{\theta}\| \|v_{\theta}\| \leq \int |f_{\theta}|^{2}. \tag{5.7}$$

Furthermore, from the definition  $2\overline{\partial}f = f_t + if_\theta$  and the inequality  $(a-2b)^2 \leq 2a^2 + 8b^2$  we obtain

$$3|f_{\theta}|^2 = 2|f_{\theta}|^2 + |f_t - 2\overline{\partial}f|^2 \le 2|df|^2 + 8|\overline{\partial}f|^2$$

Combining the previous three displayed equations and using (5.4) shows that

$$\left(1 - 4c\varepsilon^2\right) \int_A |df|^2 \le \frac{4}{3} \left(1 - 4c\varepsilon^2\right) \int_{\partial A} |df|^2.$$

Taking r small enough that  $\varepsilon = \varepsilon(r)$  satisfies  $4c\varepsilon^2 < 1/44$ , we obtain

$$\frac{2}{3}E(t) \leq E'(t).$$

Integrating this differential inequality from t to T yields (5.5).

On the cylinder  $[-T,T] \times S^1$ , each point lies in a unit disk with euclidean metric, and f satisfies the equation  $\overline{\partial} f = \nu$ . Standard elliptic estimates then bound |df| at the center point in terms of the energy in that unit disk (c.f. [PW] Theorem 2.3). Thus (5.5) implies (5.6).

In the next several sections we will repeatedly use the fact that in the cylindrical metric

$$\int \rho^{\alpha} dt \sim c_{\alpha} \rho^{\alpha} \quad \text{for } \alpha \neq 0.$$
 (5.8)

Thus, for example, (5.4) and (5.6) give

$$||f^{-1}\overline{\partial}f||_{p,A(r)} \le ||df||_{p,A(r)} \le c_p \rho^{1/3}.$$
 (5.9)

We will also use bump functions defined as follows. Fix a smooth function  $\beta : \mathbb{R} \to [0,1]$  supported on [0,2] with  $\beta \equiv 1$  on [0,1]. The function  $\beta_{\varepsilon}(z,w) = \beta(\rho/\varepsilon)$  has support where  $\rho^2 = |z|^2 + |w|^2 \le 4\varepsilon^2$ . When restricted to  $C_{\mu}$ ,  $\beta_{\varepsilon} \equiv 1$  on the 'neck' region  $A_{\mu}(\varepsilon)$  where  $\rho \le \varepsilon$ , and  $d\beta_{\varepsilon}$  is supported on two annular regions where  $\varepsilon \le \rho \le 2\varepsilon$ . We can choose  $\beta$  so that using the cylindrical metric (4.4)

$$|d\beta_{\varepsilon}| \le 2. \tag{5.10}$$

As before, we write  $f_n = (v_n, x_n, y_n)$  in coordinates centered on the image of each node.

**Definition 5.3** In the region  $\rho \leq 1$  around each node define renormalized maps  $\hat{f}_n$  by

$$\hat{f}_n = (\hat{v}_n, \hat{x}_n, \hat{y}_n) = (v_n^1 - \bar{v}_n^1, \dots v_n^k - \bar{v}_n^k, \frac{x_n}{az^s}, \frac{y_n}{bw^s})$$

where  $\bar{v}_n^i$  is the average value of  $v_n^i$  on the center circle  $\gamma_\mu = \{\rho = \sqrt{\mu}\}\$  of  $C_\mu$ .

Whenever  $\lambda_n = x_n y_n$  is non-zero  $x_n$  has no zeros and has (local) winding number s. Hence each  $\hat{x}_n$  has winding number zero, so the functions  $\log \hat{x}_n$ , and similarly  $\log \hat{y}_n$ , are well-defined. The convergence (5.2) shows that on each set  $\rho \geq r$  we have  $\hat{x}_n \to f_0/az^s = 1 + O(r)$  in  $C^1$ ; hence there is a constant c so that

$$\sup_{r \le \rho \le 1} |\log \hat{x}_n| + |\log \hat{y}_n| \le cr \qquad \forall n \ge N = N(\rho).$$
 (5.11)

**Lemma 5.4** For each sequence (5.1) we have  $\lim_{n\to\infty}\frac{\lambda_n}{\mu_n^s}=ab$ .

**Proof.** For  $G_n = \log \hat{x}_n$  the integral

$$\bar{G}_n(\rho) = \frac{1}{2\pi} \int_0^{2\pi} G_n \ d\theta$$

over the circles with fixed  $\rho$  satisfies

$$\frac{d}{dt}\bar{G}_n = \int \partial_t G_n = \int (\partial_t + i\partial_\theta)G_n = 2\int x_n^{-1} \overline{\partial} x_n d\theta.$$

To bound  $x_n^{-1} \overline{\partial} x_n$  we regard f locally as a map into  $V \times \mathbb{C}$  with its product almost complex structure  $J_V$  and a product metric  $g_V$ . Then  $\nu^N$  and  $J - J_V$  are both O(R) where  $R^2 = |x|^2 + |y|^2$ , so a slight modification of (5.4) gives  $|\overline{\partial} x_n| \leq cR |df|_{g_V}$ . Also noting that  $g = (1 + |\lambda|^2/|x|^4) g_V$  as in (4.3), we obtain

$$|x_n^{-1}\overline{\partial}x_n|^2 \le c\left(\frac{|x|^2 + |y|^2}{|x|^2}\right) |df|_{g_V}^2 = c\left(1 + \frac{\lambda^2}{|x|^4}\right) |df|_{g_V}^2 = c|df|^2.$$
 (5.12)

These equations and Lemma 5.2 give  $|\frac{d}{dt}\bar{G}_n| \leq c_1 \, \rho^{1/3}$ . Hence for  $\rho \leq r$  and n > N(r)

$$|\bar{G}_n(\rho)| \le |\bar{G}_n(r)| + c_1 \int_{\rho}^{r} \rho^{1/3} dt \le |\bar{G}_n(r)| + c_2 r^{1/3} \le c_3 r^{1/3}$$
 (5.13)

(the last inequality uses (5.11)). This implies that

$$\left| \frac{1}{2\pi} \int_{\gamma_{\mu_n}} \log \hat{x}_n \right| = \left| \bar{G}_n \left( \sqrt{|\mu_n|} \right) \right| \to 0$$

as  $n \to \infty$ . The same limit statement holds with G replaced by  $\log \hat{y}$ . For each n we can then integrate the constant

$$\log\left(\frac{\lambda_n}{ab\mu_n^s}\right) = \log\left(\frac{x_n y_n}{az^s bw^s}\right) = \log\left(\hat{x}_n \,\hat{y}_n\right)$$

over  $\gamma_{\mu_n}$  to see that

$$2\pi \log \left(\frac{\lambda_n}{ab\mu_n^s}\right) = \int_{\gamma_{\mu_n}} \log \left(\frac{\lambda_n}{ab\mu_n^s}\right) = \int_{\gamma_{\mu_n}} \log \hat{x}_n + \log \hat{y}_n \to 0.$$

The lemma follows.  $\Box$ 

**Lemma 5.5** Set  $\xi_n^V = v_n - \overline{v}_n$  and  $\xi_n^N = (x_n, y_n)$ . Then for each  $p \geq 2$  there are constants C and N = N(r) so that whenever  $\delta \leq \frac{1}{3}$ ,  $r \leq 1$  and  $n \geq N$ 

$$\int_{\rho < r} \rho^{-p\delta/2} \left( |\nabla \xi_n^V|^p + |\xi_n^V|^p + \rho^{(1-s)p} |\nabla \xi_n^N|^p + \rho^{(1-s)p} |\xi_n^N|^p \right) \le C_p r^{p/6}. \tag{5.14}$$

**Proof.** Write the annular regions  $A_{\mu}(r) = \{\rho < r\}$  in the neck as the union of annuli  $A_k = \{k \le t \le k+1\}$  of unit size and let  $\rho_k$  be the value of  $\rho$  at one end of  $A_k$ . Since  $|d\xi_n^V| = |dv_n| \le c\rho^{1/3}$  by (5.6) we have

$$\sup_{A(r)} |\xi_n^V| \le \sum_k |\operatorname{osc}_{A_k} \hat{v}| \le C \sum_k ||dv||_{4,A_k} \le C \sum_k \rho_k^{1/3} \le C r^{1/3}$$
 (5.15)

where the last inequality comes from the Riemann sum for  $\int \rho^{1/3} dt$ . Thus  $|\nabla \xi_n^V|^p + |\xi_n^V|^p \le c\rho^{p/3}$  pointwise. Integrating via (5.8) then gives the first half of (5.14).

Next, the Calderon-Zygmund inequality of [IS] shows that  $G = \log \hat{x}_n$  satisfies

$$\|dG\|_{p,A(r)} \le C \|\overline{\partial}(\beta_r G)\|_{p,A(2r)} \le C (\|d\beta_r \cdot G\|_{p,A(2r)\setminus A(r)} + \|x_n^{-1}\overline{\partial}x_n\|_{p,A(2r)})$$

We can integrate (5.12) as in (5.9), and use (5.10) and the bound (5.11) in the region  $r \le \rho \le 2r$  where  $d\beta_r \ne 0$ . These imply that the  $L^p$  norm of dG is bounded by  $c r^{1/3}$ . But then for each annulus  $A \subset A(r)$  with unit diameter we can use (5.13) and a Sobolev inequality to obtain

$$\sup_{A} \; |G| \; \leq \; |{\rm avg}_{\partial A} \; G| \; + \; {\rm osc}_{A} \; G \leq \; c \, r^{1/3} \; + \; C \, \|dG\|_{4,A(r)} \; \leq \; c \, r^{1/3} \qquad \text{for all } n \geq N(r).$$

Exponentiating this bound on G shows that  $|\hat{x}_n - 1| \leq cr^{1/3}$  in A(r), and that in turn gives  $|d\hat{x}_n| = |\hat{x}_n dG| \leq c |dG|$ . Consequently  $\xi_n^x = \hat{x}_n(az^s)$  satisfies

$$|\xi_n^x| \le c\rho^{s+1/3}$$
 and  $|d\xi_n^x| \le c\rho^s (1 + |dG|)$  (5.16)

(after noting that |dz/z| is bounded). Of course the same bounds hold for the y components, so integration, combined with the  $L^p$  bound on dG, gives the second half of (5.14).

## 6 The Space of Approximate Maps

The limit argument of section 3 shows that as  $\lambda \to 0$  holomorphic maps  $f_{\lambda}$  into  $Z_{\lambda}$  converge to maps into  $X \cup Y$  with matching conditions along V, i.e. to a maps in  $\mathcal{M}_{s}^{V}(X) \times_{ev} \mathcal{M}_{s}^{V}(Y)$ . Lemma 5.5 gives further information about the convergence near the matching points; it shows

that for small  $\lambda$  the maps  $f_{\lambda}$  are closely approximated by maps  $g(z, w) = (\overline{v}, az^s, bw^s)$  in local coordinates. Over the next four sections we will reverse this process, showing how one can use  $\mathcal{M}_s \times_{ev} \mathcal{M}_s$  to construct a model  $\mathcal{AM}_s(\lambda)$  for the space of stable maps into  $Z_{\lambda}$ . The final result is stated as Theorem 10.1.

The construction has two main steps. In the first, maps f in a compact set  $\mathcal{K} \subset \mathcal{M}_s^V(X) \times_{ev} \mathcal{M}_s^V(Y)$  are smoothed in a canonical way to construct maps F into  $Z_{\lambda}$  which are nearly holomorphic. The second step corrects those approximate maps F to make them truly holomorphic. This section describes the canonical smoothing and the resulting space of approximate maps and introduces norms on the space of maps which capture the convergence of the renormalized maps. Those norms lead to a precise statement that the approximate maps are nearly  $(J, \nu)$ -holomorphic.

The maps alone cannot be canonically smoothed — more data are needed. This harks back to the comment at the end of Section 3 that each f will generally be the limit of many maps into  $Z_{\lambda}$ . Recall that  $f \in \mathcal{M}_{s}^{V}(X) \times_{ev} \mathcal{M}_{s}^{V}(Y)$  is a map from an  $\ell(s)$ -nodal curve  $C_{0}$  whose nodes  $x_{k} = y_{k}$  are mapped into V with contact of order  $s_{k}$ . As in section 3  $C_{0}$  has an  $\ell$  dimensional family of smoothings  $C_{\mu}$ ,  $\mu = (\mu_{1}, \dots, \mu_{\ell})$ . Lemma 5.4 shows that  $C_{0}$  is the limit of maps into  $Z_{\lambda}$  only if  $\mu$  satisfies  $a_{k}b_{k}\mu_{k}^{s_{k}} = \lambda$ . That leaves  $|s| = s_{1}s_{2}\cdots s_{\ell}$  possibilities for  $\mu$  corresponding to the different choices of root for each  $\mu_{k}$ . Thus the maps into  $Z_{\lambda}$  near f are specified by pairs  $(f, \mu)$ , with the  $\mu$  specifying the deformation of the domain.

Globally, we have  $\lambda \in N_X \otimes N_Y \cong \mathbb{C}$  (via a fixed trivialization), so (5.3) implies that at each node the coefficients  $a_k, b_k$  determine a section

$$\frac{\lambda}{a_k b_k} \in \Gamma\left(\mathcal{L}_k^* \otimes (\mathcal{L}_k')^*\right)$$

over  $\mathcal{M}_s^V(X) \times_{ev} \mathcal{M}_s^V(Y)$ . The  $s_k^{th}$  root of this section is a multisection of  $\mathcal{L}_k^* \otimes (\mathcal{L}_k')^*$ ; considering all k at once defines a multisection of the direct sum of the  $\mathcal{L}_k^* \otimes (\mathcal{L}_k')^*$ . This gives an intrinsic model for our space  $\mathcal{A}\mathcal{M}_s(\lambda)$  of approximate maps:

**Definition 6.1** For each s and  $\lambda \neq 0$ , the model space  $\mathcal{AM}_s(\lambda)$  is the multisection of

$$\bigoplus_{k=1}^{\ell} \left[ \mathcal{L}_k^* \otimes (\mathcal{L}_k')^* \right] \to \mathcal{M}_s^V(X) \times_{ev} \mathcal{M}_s^V(Y)$$

consisting at  $f_0$  of those  $\mu = (\mu_1, \dots, \mu_\ell)$  which satisfy

$$\mu_k^{s_k} = \frac{\lambda}{a_k b_k} \quad \text{for each } k. \tag{6.1}$$

This model space is an |s|-fold cover of  $\mathcal{M}_s^V(X) \times_{ev} \mathcal{M}_s^V(Y)$ , and hence is a manifold for generic  $(J, \nu)$ . Elements of the model space are pairs  $(f, \mu)$  where  $f: C_0 \to Z_0$  and  $\mu$  satisfies (6.1). Each such element gives rise to an approximate holomorphic map as follows.

**Definition 6.2** For each  $(f, \mu) \in \mathcal{AM}_s(\lambda)$ ,  $\lambda \neq 0$ , define an approximate holomorphic map  $F = F_{f,\mu} : C_{\mu} \to Z_{\lambda}$  by

$$F = f - \sum \beta_k (f - p_k) \tag{6.2}$$

where  $\beta_k$  is bump function (5.10) with  $\varepsilon = |\lambda|^{1/4s_k}$  in coordinates  $(z_k, w_k)$  around the  $k^{\text{th}}$  node,  $p_k$  is the image of the node in those coordinates, and f is the restriction of f(z, w) = (v(z, w), x(z), y(w)) to  $C_{\mu}$ .

Altogether, the association  $(f, \mu) \mapsto (F_{f,\mu}, C_{\mu})$  defines a 'gluing map'

$$\Gamma_{\lambda} : \mathcal{AM}_s(\lambda) \to \operatorname{Maps}_s(C, Z_{\lambda} \times \mathcal{U})$$
 (6.3)

This map is injective: if  $\Gamma_{\lambda}(f,\mu) = \Gamma_{\lambda}(f',\mu')$  then f and f' are  $(J,\nu)$ -holomorphic maps which agree on the set where  $\rho > 1$  and therefore, by the unique continuation property of elliptic equations, agree everywhere.

In section 9 we will show that  $\Gamma_{\lambda}$  is a diffeomorphism onto a submanifold. Here, as a preliminary, we introduce norms which make the space of maps in (6.3) into a Banach manifold.

**Norms**. We will use weighted Sobolev norms tailored for our problem. On the domain we continue to use the cylindrical metric (4.4) and to use (4.5) to measure distance between curves. In the target we identify a neighborhood of V in Z with the disk bundle of the bundle  $N_X \oplus N_Y$  over V; in that neighborhood we can then decompose vector fields  $\xi$  into components  $(\xi^V, \xi^x, \xi^y)$  where  $\xi^V$  is horizontal with respect to the connection on  $N_X \oplus N_Y$  and the  $\xi^x$  and  $\xi^x$  are tangent to the fibers of  $N_X$  and  $N_Y$ .

In a neighborhood of each node  $p_k$  let  $\gamma_k$  be the circle  $\rho = \sqrt{|\mu|}$  in  $C_{\mu}$  and let  $\beta_k$  be as in (6.2). Given  $\xi$ , subtract the (extended) average value of the V components, defining

$$\zeta = \xi^V - \beta_k \overline{\xi} \quad \text{where} \quad \overline{\xi} = \frac{1}{2\pi} \int_{\gamma_k} \xi^V \in T_{p_k} V.$$
 (6.4)

These averaged vectors  $\overline{\xi}_k$  at the different nodes can be assembled into a single vector  $\overline{\xi} \in T_p V^{\ell}$  where  $p = (p_1, \dots, p_{\ell})$ . Similarly, the  $(\zeta_k, \xi_k^x, \xi_k^y)$  extend to a global vector field on  $C_{\mu}$ 

$$\zeta = \xi - \sum \beta_k \bar{\xi}_k \tag{6.5}$$

where  $\beta_k$  is the bump function (5.10) with  $\varepsilon = 1$  centered on the node  $p_k$ . Fix  $\delta > 0$  and set

$$\|\zeta\|_{1,p,s}^p = \int_{C_u} \rho^{-\delta p/2} |\nabla(W\zeta)|^p + |W\zeta|^p$$
(6.6)

where  $\nabla$  is the covariant derivative of the cylindrical metric on the domain and the metric induced on  $Z_{\lambda}$  from Z while the endomorphism  $W: \zeta \to W\zeta$  weights the normal components around each node:

$$W\zeta = (1 - \sum \beta_k)\zeta + \sum \beta_k \left(\zeta^V, \frac{\zeta_k^x}{z^{s_k - 1}}, \frac{\zeta_k^y}{w^{s_k - 1}}\right). \tag{6.7}$$

Here a complex valued function  $\phi = u + iv$  acts on (real) vector field  $\zeta$  by  $\phi \cdot \zeta = u \cdot \zeta + v \cdot J\zeta$ , with  $|\phi\zeta| = |\phi| \cdot |\zeta|$ .

**Definition 6.3** Given a tangent vector  $(\xi, h)$  to the space  $Map_s(C, Z_\lambda \times \mathcal{U})$ , we form the triple  $(\zeta, \bar{\xi}, h)$  as in (6.5) and define the weighted  $L^1_s$  norm

$$\|(\xi,h)\|_{1} = \|\zeta\|_{1,2,s} + \|\zeta\|_{1,4,s} + |\bar{\xi}| + \|h\|$$

$$(6.8)$$

where ||h|| is given by (4.6). For 1-forms  $\eta \in \Omega^{0,1}(f_{\mu}^*TZ_{\lambda})$  we do the same without averaging:

$$\|\eta\|_{1} = \|\eta\|_{1,2,s} + \|\eta\|_{1,4,s}. \tag{6.9}$$

The weighted  $L_s^0$  norm  $\|\cdot\|_0$  and the weighted  $L_s^2$  norm  $\|\cdot\|_2$  are defined similarly.

The norm  $\|(\xi,h)\|_1$  dominates the  $C^0$  norm (since  $L^{1,4} \hookrightarrow C^0$ ). Hence we can use it to complete the space of  $C^{\infty}$  maps, making  $\operatorname{Map}_s(C, Z_{\lambda} \times \mathcal{U})$  a Banach manifold with neighborhoods modeled by  $(\xi,h)$ .

**Remark 6.4** Note that the norms defined above make sense also at  $\lambda = 0$ , where the average value  $\overline{\xi}$  is equal to the value of  $\xi$  at the double points  $x_k, y_k$ .

We conclude this section by showing that the approximate maps are nearly holomorphic. The specific statement is that the quantity  $\overline{\partial}F - \nu_F$ , which measures the failure of the approximate map to be  $(J, \nu)$ -holomorphic, is small in the norms just introduced.

**Lemma 6.5** For  $\delta \leq \frac{1}{3}$  and  $\lambda$  sufficiently small, each  $F = F_{f,\mu}$  satisfies  $\|\overline{\partial}F - \nu\|_0 \leq c |\lambda|^{1/6|s|}$ .

**Proof.** Let  $N_k(\mu)$  be the region around the node  $p_k$  where  $\rho \leq 2\sqrt[4]{|\mu|}$ . Outside  $\bigcup_k N_k(\mu)$   $F \equiv f$  is  $(J,\nu)$ -holomorphic and therefore  $\Phi = \overline{\partial}F - \nu_F$  vanishes. When  $\lambda \sim \mu^s$  is sufficiently small the image of each  $N_k(\mu)$  lies in a neighborhood of V where we can separate components tangent and normal to V. Taking  $F = f - \beta(f - p)$ ,

$$\Phi(z) = (1-\beta)\overline{\partial}_{J_F} f + d\beta \left[ (f-p) - J_F(f-p)j \right] - \nu(z, F(z))$$

where  $J_F$  means J at the point F(z). Since f is  $(J_f, \nu_f)$ -holomorphic and  $|d\beta|$  is bounded

$$|W\Phi| \le |W((J_F - J_f)df)| + c|W(f - p)| + |W(\nu_F - \nu_f)|.$$

Now the local expansions of x(z) and y(z) show that  $|W(f-p)| \approx |(v-v_0,az,bw)| \leq c\rho$  and similarly  $|W((J_F-J_f)df)| \leq c\rho$ . Because  $\nu^N$  vanishes along V, the normal component of  $\nu_F - \nu_f$  is bounded by  $c|F^N| \leq c\rho^s$ , while  $|(\nu_F - \nu_f)^V| \leq c|F-f| \leq c\rho$ . Thus  $W\Phi \leq c\rho$ . Integrating over  $N_k(\mu)$  using (5.8) then gives

$$||W\Phi||_{0,p,s;N_k(\mu)}^p \le c |\mu_k|^{\frac{p}{6}} \le c |\lambda|^{p/(6s_k)}.$$

The lemma follows by summing on k and on p = 2, 4.  $\square$ 

#### 7 Linearizations

This section describes the linearization of the  $(J, \nu)$ -holomorphic map equation as an operator on the Sobolev spaces of Definition 6.3. We do this first for the space  $\mathcal{M}_s^V(X)$  which defines the relative invariants, then for the space  $\mathcal{M}_s(Z_0) = \mathcal{M}_s^V(X) \times_{ev} \mathcal{M}_s^V(Y)$  of maps into the singular space  $Z_0$ . That serves as background for our main purpose: describing the linearization operator  $\mathbf{D}_{\mu}$  at an approximate map into  $Z_{\lambda}$  and its adjoint  $\mathbf{D}_{\mu}^*$ . These are the operators that will be used in Section 9 to correct the approximate maps into holomorphic maps.

To begin with, let  $(f,j) \in \mathcal{M}_s^V(X)$ , and consider the linearization (1.6) with the norms (6.8) and (6.9). It is convenient to decompose  $\xi$  into  $\zeta$  and  $\overline{\xi}$  as in (6.5) and to consider this linearization as the operator

$$\mathbf{D}_{(f,j)}: L_s^1(\Lambda^0(f^*TX)) \oplus T_p V^{\ell} \oplus T_C \mathcal{M}_{g,n+\ell} \to L_s(\Lambda^1(f^*TX))$$
(7.1)

defined in terms of the operator (1.6) by

$$\mathbf{D}_{(f,j)}(\zeta,\overline{\xi},h) = D_{(f,j)}\left(\zeta + \sum \beta_k \overline{\xi}_k, h\right). \tag{7.2}$$

Note that if  $(\xi, h)$  satisfies  $D_f(\xi, h) = 0$  then the condition that  $\xi^N$  has a zero of order s at x is equivalent to  $\|\xi\|_1 < \infty$ . One of the implications is obvious, while the other follows from  $\|\rho^{-2\delta}W\zeta\|_{L^4} \leq \|\xi\|_1 < \infty$  by basic elliptic estimates. This means that with the norms above the domain of  $\mathbf{D}_f$  includes already the linearization of the contact conditions. Moreover, for generic  $0 < \delta < 1$ ,  $\mathbf{D}_f$  is a bounded linear Fredholm operator with respect to these norms and models the space  $\mathcal{M}_s^V(X)$ . In particular, for generic V-compatible  $(J, \nu)$   $\mathcal{M}_s^V(X)$  is an orbifold of dimension  $2 \operatorname{ind}_{\mathbb{C}} \mathbf{D}_f$  and we have the following two facts.

**Lemma 7.1** (a)  $st \times ev : \mathcal{M}_s^V(X) \to \mathcal{M}_{g,n+\ell} \times V^{\ell}$  is a smooth map of Banach manifolds.

- (b) The 'leading coefficient map' (5.3) defines a smooth section of the bundle  $\bigoplus_{k=1}^{\ell} \mathcal{L}_k \otimes \mathcal{N}_X$  over  $\mathcal{M}_{\mathfrak{s}}^V(X)$ .
- **Proof.** (a) It suffices to show that the linearization is a smooth map everywhere. Let  $(\xi, h)$  be a tangent vector to  $\mathcal{M}_s^V(X)$  and decompose  $\xi = \zeta + \beta \overline{\xi}$  with  $\overline{\xi} = \xi(x) \in T_pV$ . The linearization of the map  $st \times ev$  is  $(\xi, h) \to (h, \overline{\xi})$  which is obviously smooth with our norms.
- (b) Choose a path  $(f_t, j_t)$  in  $\mathcal{M}_s^V$  and let  $(\xi, k)$  be its tangent vector at t = 0. Assume for simplicity that  $\ell = 1$  and let  $f_t^N = a_t z^s + O(|z|^{s+1})$  be the expansion near the single point x with  $f(x) \in V$ . Writing  $\xi = \zeta + \beta \overline{\xi}$  with  $\overline{\xi} = \xi(x) \in T_p V$  and differentiating, we see that the tangential component  $\dot{\zeta} \in TV$  vanishes and the normal component is  $\dot{\zeta}^N = \dot{a}_t z^s + O(|z|^{s+1})$ . Since  $D_f(\xi, h) = 0$ , elliptic bootstraping gives  $|\dot{a}_t| \leq |\dot{\zeta}^N|_{C^s} \leq c ||(\xi, h)||_1$ .

For maps  $f_0$  into  $Z_0$  the linearization of the  $(J, \nu)$ -holomorphic map equation has a form similar to (7.1), as follows. Thinking of  $f_0$  as a pair of maps  $(f_1, f_2) \in \mathcal{M}_s^V(X) \times_{\text{ev}} \mathcal{M}_s^V(Y)$ , a variation  $\xi$  of  $f_0$  consists of continuous sections on each component of the domain which have the same value on both sides of each node. This means that the domain of  $D_0$  consists of sections  $\xi = (\zeta_1, \overline{\xi}_1, h_1; \zeta_2, \overline{\xi}_2, h_2)$  with the matching condition  $\overline{\xi}_1 = \overline{\xi}_2$  at each node in V. The corresponding operator  $\mathbf{D}_0$  whose kernel models  $T_{f_0}\mathcal{M}_s(Z_0) = T_{(f_1, f_2)}\mathcal{M}_s^V(X) \times_{\text{ev}} \mathcal{M}_s^V(Y)$  is

$$\mathbf{D}_0: L_s^{1,p}(\Lambda^0(f_0^*TZ_0)) \oplus T_pV^{\ell} \oplus T_{C_1}\widetilde{\mathcal{M}} \oplus T_{C_2}\widetilde{\mathcal{M}} \to L^p(\Lambda^1(f_0^*TZ_0))$$

$$\tag{7.3}$$

where  $\Lambda^i(f_0^*TZ_0)$ ) means  $\Lambda^i(f_1^*TX)$ )  $\oplus \Lambda^i(f_2^*TY)$ ). Again, one can verify that the evaluation map  $ev: \mathcal{M}_s^V \times \mathcal{M}_s^V \to V \times V$  is smooth and its image is tranverse to the diagonal  $\Delta$  for generic  $(J, \nu)$ . Thus generically Coker  $\mathbf{D}_0 = 0$  and the space  $\mathcal{M}_s^V \times_{ev} \mathcal{M}_s^V = ev^{-1}(\Delta)$  is a smooth orbifold as in Lemma 3.5.

The space of stable maps is defined as the set of  $(J, \nu)$ -holomorphic maps modulo diffeomorphisms. The compute a linearization, we choose a path in the moduli space, lift to a local slice to the action of the diffeomorphism group, and differentiate. In fact the constructions of Section 4 provide such a local slice at the approximate maps of Definition 6.2. We will describe the slice, then use it to compute the linearization operator.

Recall that a  $(J, \nu)$ -holomorphic map is a pair  $(f, \phi): \Sigma \to Z_{\lambda} \times \overline{\mathcal{U}}_{g,n}$  where  $\Sigma$  is a smooth 2-manifold and  $\phi$  identifies  $\Sigma$  with a fiber of the universal curve  $\overline{\mathcal{U}}_{g,n}$  and where f satisfies the

 $(J,\nu)$ -holomorphic map equation with respect to that complex structure. Given an approximate map  $(F,\phi): \Sigma \to Z_{\lambda}$  we can construct 1-parameter families of deformations  $(F_t,\phi_t)$  as follows. Fix a section  $\xi$  of  $F^*TZ_{\lambda}$  over  $\Sigma$  and a vector  $v \in T_C\overline{\mathcal{M}}_{g,n}$  tangent to the space of stable curves at  $C_{\mu} = \phi(\Sigma)$ . As in section 4, the path  $\mu_t = \exp(tv)$  lifts to a path of diffeomorphisms  $\exp(t\tilde{v}): C_{\mu} \to C_{\mu_t}$ . This gives the family of maps

$$(F_t, \phi_t) = (\exp_F \xi, \ \phi \circ \exp^{-1}(t\tilde{v})) \tag{7.4}$$

and a path  $j_t = [\exp(t\tilde{v})]^*j$  of complex structures with initial tangent  $h_v \in \Omega^{01}(TC)$ . By Lemma 4.2 the norm (4.6) of this h is uniformly equivalent to  $|\dot{\mu}|$ .

That understood, the linearization at the approximate map  $F_{\mu} = F_{f,\mu}$  is then given by (1.6) with  $h = h_v$  as above. As before, the decomposition (6.5) of  $\xi$  into  $\zeta$  and  $\overline{\xi}$  allows us to consider the linearization as the operator

$$\mathbf{D}_{\mu}: L_{s}^{1}(\Lambda^{0}(F_{\mu}^{*}TZ_{\lambda})) \oplus T_{p}V^{\ell} \oplus T_{C}\mathcal{M}_{g,n} \to L_{s}(\Lambda^{1}(F_{\mu}^{*}TZ_{\lambda}))$$

$$(7.5)$$

defined by (7.2) with (f,j) replaced by the approximate map  $F_{\mu}$ .

**Lemma 7.2** For generic  $0 < \delta < 1$ , (7.5) is a bounded Fredholm operator for each approximate map  $F_{\mu}$  with

index<sub>C</sub> 
$$\mathbf{D}_{\mu} = \frac{1}{2} \dim \mathcal{M}_{s}^{V}(X) \times_{ev} \mathcal{M}_{s}^{V}(Y)$$

**Proof.** Combining (7.5) and (1.7) and noting that  $|df| \le c\rho$  and gives the pointwise bound

$$|\mathbf{D}(\zeta, a, h)| = |L(\zeta) + aL(\beta) + Jdfh| \le c(|\nabla \zeta| + |\zeta| + |a| + \rho|h|).$$

It also shows that the normal component near each node is

$$[\mathbf{D}(\zeta, a, h)]^{N} = \overline{\partial}\zeta^{N} + (\nabla_{\zeta^{N}}J)^{N} \circ df \circ j + (\nabla_{\zeta^{V} + a}J)^{N} \circ df^{N} \circ j + J \circ df^{N}(h) + O(\rho)$$

(the missing term  $(\nabla_{\zeta^V+a}J)^N\circ df^V$  vanishes at  $x_i$  because a is tangent to V and V is J-holomorphic). Because  $\nabla J$  is bounded and  $df^N=s(z^{s-1}dz,w^{s-1}dw)+O(\rho^s)$  this gives

$$|W\mathbf{D}(\zeta, a, h)^N| \le c (|W\overline{\partial}\zeta^N| + |W\zeta| + |a| + \rho|h|).$$

Differentiating  $W^{-1}\zeta^N=(z^{s-1}\zeta^x,w^{s-1}\zeta^y)$  and noting that  $|dz|,|dw|\leq\rho$  shows that  $|W\overline{\partial}\zeta|\leq c(|\nabla W\zeta|+|W\zeta|)$ . Hence we have the pointwise bound

$$|W\mathbf{D}(\zeta, a, h)| < c(|\nabla(W\zeta)| + |W\zeta| + |a| + \rho|h|).$$

Integrating and using the Sobolev embedding on the h shows that  $\mathbf{D}$  is bounded as stated. The fact that  $\mathbf{D}$  is Fredholm follows from [Lo].  $\Box$ 

The adjoint  $\mathbf{D}^*$  of (7.5) with respect to the weighted  $L^2$  norms is determined by the relation

$$\langle (\zeta, a, h), \mathbf{D}^* \eta \rangle = \langle \mathbf{D}(\zeta, a, h), \eta \rangle$$

Fixing the map F and putting in  $\mathbf{D}(\zeta, a, h) = L(\zeta + \beta a) + JF_*h$  as in (1.7) one finds that

$$\mathbf{D}^* \eta = D^* \eta + A \eta + B \eta \quad \text{where} \quad \begin{cases} D^* \eta = \left( \rho^{-\delta} W \overline{W} \right)^{-1} L^* (\rho^{-\delta} W \overline{W} \eta) \\ A \eta = \int_{C_{\mu}} \rho^{-\delta} \left[ (\overline{\partial} \beta) \eta^V + \beta \langle \nabla J \circ df \circ j, \eta^V \rangle \right] \\ B \eta = -F_*^t J \eta. \end{cases}$$
(7.6)

where W is given by (6.7) (and  $\overline{W}$  is the corresponding weighting by  $\overline{z}$  and  $\overline{w}$ ),  $F_*^t$  is the transpose of the differential of F and  $L^*$  is the  $L^2$  adjoint of operator L of (1.8)

$$L^* \eta = \partial_f^* \eta + S^* \eta + T^* \eta \tag{7.7}$$

where  $S^*$  and  $T^*$  are the adjoints of S and T, and  $\partial_f^* = -\sigma_J^* \circ \nabla$  where  $\sigma_J^*$  is the adjoint of the symbol of  $\overline{\partial}_f$ .

### 8 The Eigenvalue Estimate

We now come to the key analysis step: obtaining estimates on the linearization D of the  $(J, \nu)$ holmorphic map equation along the space of approximate maps. We establish a lower bound for
the eigenvalues of  $DD^*$  and construct a right inverse P for  $D^*$ . This operator P will be used in
the next section to correct approximate maps to true holomorphic maps.

To get uniform estimates we fix  $(J, \nu)$  generic in the sense of Lemma 3.5. We continue to work with  $\delta$ -flat maps, which we will call  $\delta_0$ -flat in this section to avoid confusion with the exponential weight  $\delta$  of the norm (6.6), which will also appear. As in (3.11) this  $\delta_0$  defines a compact set  $\mathcal{K}_{\delta_0} \subset \mathcal{M}_s^V(X) \times_{ev} \mathcal{M}_s^V(Y)$  of (3.11) and corresponding subsets

$$\mathcal{A}\mathcal{M}_{\lambda}^{\delta_0} \subset \mathcal{A}\mathcal{M}_{\lambda} \quad \text{and} \quad \mathcal{A}_{\lambda}^{\delta_0} \subset \mathcal{A}_{\lambda}$$
 (8.1)

of the model space and the space of approximate maps. Thus  $\mathcal{AM}_{\lambda}^{\delta_0}$  is the inverse image of  $\mathcal{K}_{\delta_0}$  under the covering map of Definition 6.1 and  $\mathcal{A}_{\lambda}^{\delta_0}$  is the image of  $\mathcal{AM}_{\lambda}^{\delta_0}$  under the gluing map (6.3). For the maps  $(f_1, f_2)$  in  $\mathcal{K}_{\delta_0}$  the leading coefficients  $|a_k|$ ,  $|b_k|$  at the nodes are uniformly bounded away from 0 and  $\infty$ , and therefore  $|\lambda|$  is uniformly equivalent to  $|\mu_k|^{s_k}$  for each k. That understood, the aim of this section is to prove the following analytic result.

**Proposition 8.1** There are constants E, c > 0 independent of  $\lambda$  and of  $f \in \mathcal{K}_{\delta_0} \subset \mathcal{M}_s^V \times_{ev} \mathcal{M}_s^V$  such that the linearization  $D_{\mu}$  at an approximate map  $F = F_{f,\mu}$  has a partial right inverse

$$P_{\mu}: L^0_s(\Lambda^{0,1}(F^*TZ_{\lambda})) \to L^1_s(\Lambda^0(F^*TZ_{\lambda})) \oplus T_{C_{\mu}}\mathcal{M}$$

such that

$$c\|\eta\|_0 \le \|P_\mu \eta\|_1 \le E^{-1}\|\eta\|_0 \tag{8.2}$$

**Proof.** By the spectral theorem for elliptic operators, the domain of  $D^*$  decomposes as the direct sum of finite-dimensional eigenspaces of  $D^*D$  and the target similarly decomposes into eigenspaces of  $DD^*$ . The eigenvalues are non-negative and the eigenfunctions are smooth. Lemma 8.4 below shows that there is a uniform lower bound E on the first eigenvalue of  $D_{\mu}D_{\mu}^*$  for

approximate maps  $F_{\mu}$ . Using that, Lemma 8.5 shows that  $D_{\mu}D_{\mu}^{*}$  is uniformly invertible. Therefore  $P_{\mu} = D_{\mu}^{*}(D_{\mu}D_{\mu}^{*})^{-1}$  is a partial right inverse for  $D_{\mu}$  that satisfies the required estimate.

Let  $N_k$  be the neck region defined by  $\rho \leq 1/k$ . We start by proving the following essential estimate:

**Proposition 8.2** For  $\delta > 0$  small there are constants  $k_0$  and c such that for all  $\lambda$  sufficiently small, all approximate maps  $F \in \mathcal{A}_{\lambda}^{\delta_0}$  and each neck  $N_k$  with  $k \geq k_0$ , each  $\eta \in \Omega^{0,1}(F_{\mu}^*TZ_{\lambda})$  satisfies

$$\int_{N_k} \rho^{\delta} \left( |\nabla \eta|^2 + |\eta|^2 \right) \le c \int_{N_k} \rho^{\delta} |L^* \eta|^2 + c \int_{\partial N_k} \rho^{\delta} \left( |\nabla \eta|^2 + |\eta|^2 \right). \tag{8.3}$$

**Proof.** For  $\delta > 0$  write  $\rho^{\delta}$  as the derivative of  $\psi(t) = \int_0^t \rho^{\delta}(\tau) d\tau$  and integrate by parts:

$$\int_{N} \rho^{\delta} |\eta|^{2} = \int_{N} \psi' |\eta|^{2} dt d\theta \leq \int_{N} |\psi| \cdot 2\langle \eta, \nabla \eta \rangle + \int_{\partial N} |\psi| \cdot |\eta|^{2}.$$

Because  $\rho^2 = 2|\mu|\cosh(2t)$  satisfies  $\rho^2 \leq 2|\mu|e^{2t} \leq 2\rho^2$ , we have  $|\psi| \leq c\rho^{\delta}/|\delta|$ , so the first integrand on the right is bounded by  $\frac{1}{2}|\eta|^2 + c_{\delta}|\nabla\eta|^2$ . Rearranging gives

$$\int_{N} \rho^{\delta} |\eta|^{2} \leq c \int_{N} \rho^{\delta} |\nabla \eta|^{2} + c \int_{\partial N} \rho^{\delta} |\eta|^{2}. \tag{8.4}$$

Now on the cylinder  $N_k$  every (0,1) form  $\eta$  can be written  $\eta = \eta_1 dt - (J\eta_1) d\theta$  where  $\eta_1$  is a section of the pullback tangent bundle. Denote by  $U = F_*\partial_t$  and  $V = F_*\partial_\theta$ . Note that in the usual coordinates both  $|\nabla J|$  and  $|\nabla \nu|$  are bounded, so when translating into cylindrical coordinates on the domain we get  $|dF| \leq c\rho$  and thus

$$L^*\eta = -\nabla_U \eta_1 + J\nabla_V \eta_1 + O(\rho|\eta|)$$

Therefore

$$c(|\rho\eta|^2 + \rho |\eta| |\nabla\eta|) + |L^*\eta|^2 \geq |\nabla_U\eta_1|^2 + |J\nabla_V\eta_1|^2 - 2\langle\nabla_U\eta_1, J\nabla_V\eta_1\rangle$$
  
$$\geq \frac{1}{2}|\nabla\eta|^2 - 2\langle\nabla_U\eta_1, J\nabla_V\eta_1\rangle$$

Next, differentiating the 1-form  $\omega = \langle \eta_1, J\nabla_U \eta_1 \rangle dt + \langle \eta_1, J\nabla_V \eta_1 \rangle d\theta$  and moving J past  $\nabla$ 

$$d\omega = (2\langle \nabla_{U}\eta_{1}, J\nabla_{V}\eta_{1}\rangle + \langle \eta_{1}, \nabla_{U}(J\nabla_{V}\eta_{1}) - \nabla_{V}(J\nabla_{U}\eta_{1})\rangle) dtd\theta$$

$$\geq 2\langle \nabla_{U}\eta_{1}, J\nabla_{V}\eta_{1}\rangle + \langle \eta_{1}, J\mathcal{R}(U, V)\eta_{1}\rangle - c|\nabla J||dF||\eta||\nabla \eta|$$

$$\geq 2\langle \nabla_{U}\eta_{1}, J\nabla_{V}\eta_{1}\rangle + \langle \eta_{1}, J\mathcal{R}(U, V)\eta_{1}\rangle - c\rho|\eta||\nabla \eta|$$

where  $\mathcal{R}$  is the curvature of  $\nabla$ . Combining the last two displayed equations, multiplying by  $\rho^{\delta}$ , integrating by parts and using the bound  $2\rho|\eta| |\nabla \eta| \leq \rho |\nabla \eta|^2 + \rho |\eta|^2$  then gives

$$\frac{1}{2} \int_{N_k} \rho^{\delta} |\nabla \eta|^2 \leq \int_{N_k} \rho^{\delta} \left[ |L^* \eta|^2 + \langle \mathcal{R}(U, V) \eta_1, J \eta_1 \rangle \right] - d \left( \rho^{\delta} \right) \wedge \omega + \int_{\partial N_k} \rho^{\delta} \omega \qquad (8.5)$$

$$+ c \int_{N_k} \rho^{\delta+1} (|\eta|^2 + |\nabla \eta|^2)$$

Because the domain metric is flat,  $\mathcal{R}$  is the curvature of  $Z_{\lambda}$ . By the Gauss equations

$$\langle \mathcal{R}(U,V)\eta_1,J\eta_1\rangle = \langle R^Z(U,V)\eta_1,J\eta_1\rangle - \langle h(\eta_1,V),h(J\eta_1,U)\rangle + \langle h(J\eta_1,V),h(\eta_1,U)\rangle$$

where  $R^Z$  is the curvature of Z and h is the second fundamental form of  $Z_{\lambda} \subset Z$ , which satisfies  $|h(F_*v,\cdot)| \leq c|v|$  for any v. Since  $R^Z$  is bounded then the term containing it is dominated by  $c\rho^2|\eta|^2$ . Also, as in Lemma 6.5  $|V-JU|=|\overline{\partial}F|\leq c\rho$ . Hence we can replace V by JU with small error:

$$\langle \mathcal{R}(U, V)\eta_1, J\eta_1 \rangle \leq -\langle h(\eta_1, JU), h(J\eta_1, U) \rangle + \langle h(J\eta_1, JU), h(\eta_1, U) \rangle + c\rho |\eta|^2. \tag{8.6}$$

Observe that if we had  $\nabla J = 0$  along  $Z_{\lambda}$  then h would be linear in J and the two h terms above would reduce to  $-2|h(\eta_1, U)|^2 \leq 0$ . In our case the  $\nabla J$  term is of order  $\rho|\eta|$  therefore

$$\langle \mathcal{R}(U,V)\eta_1, J\eta_1 \rangle \leq c\rho |\eta|^2.$$
 (8.7)

which can be absorbed in the last term of (8.5).

It remains to bound the  $\omega$  term in (8.5). As in (4.3) we can introduce cylindrical coordinates  $\tau = \log |x/\lambda|$  and  $\Theta$  on  $N_X$  and normal (Fermi) coordinates in the V direction. Then the metric on  $Z_{\lambda}$  is  $R^2(d\tau^2 + d\Theta^2) + g^V$  where  $R^2 = |x|^2 + |y|^2 = 2|\lambda|\cosh(2\tau)$  and g is the metric of V. The formula for F shows that in this basis  $F_*\partial_{\theta} = s\partial_{\alpha} + O(\rho)\partial_i$  and a computation shows that the Christoffel symbols are all bounded and those in the  $\Theta$  direction are

$$\Gamma_{\Theta\Theta}^{\Theta} = \Gamma_{\tau\tau}^{\Theta} = 0 \quad \Gamma_{\Theta\tau}^{\Theta} = -\Gamma_{\Theta\Theta}^{\tau} = \tanh(2\tau).$$

Thus  $\nabla_{\theta} = \partial_{\theta} + \tanh(2\tau)J + A\rho$  where A is bounded. Recalling the definition of  $\rho^2$  from (4.4), we have

$$-d(\rho^{\delta}) \wedge \omega = -\partial_t \rho^{\delta} \langle \eta_1, J \nabla_V \eta_1 \rangle dt d\theta = \delta \rho^{\delta} \tanh(2t) \langle J \eta_1, \nabla_V \eta_1 \rangle dt d\theta$$
 (8.8)

Because  $g_{\lambda}$  is independent of  $\theta$  in these coordinates, using the same methods as in (5.7) combined with the fact that  $|\tanh(2\tau)| \leq 1$  we get the bound

$$-\tanh(2\tau)\int_{S^1}\langle J\eta_1,\partial_\theta\eta_1\rangle d\theta \leq \int_{S^1}|\partial_\theta\eta_1|^2 d\theta$$

Moving all the terms on the same side we get

$$0 \le \int_{S^1} \langle \nabla_{\theta} \eta_1, \partial_{\theta} \eta_1 \rangle + c\rho \int_{S^1} (|\eta|^2 + |\nabla \eta|^2)$$

which the implies

$$\tanh(2\tau) \int_{S^1} \langle J\eta_1, \nabla_\theta \eta_1 \rangle d\theta \leq \int_{S^1} |\nabla_\theta \eta_1|^2 d\theta + c\rho \int_{S^1} (|\eta|^2 + |\nabla \eta|^2)$$

But  $\tanh(2\tau) = \tanh(2st) + O(\rho)$  and  $0 \le \tanh(2t)/\tanh(2st) \le 1$  so combining the last displayed equation with (8.8) gives

$$-\int_{N_k} d(\rho^{\delta}) \wedge \omega \le \delta \int_{N_k} \rho^{\delta} |\nabla \eta|^2 + c \int_{N_k} \rho^{1+\delta} (|\nabla \eta|^2 + |\eta|^2)$$

Inserting this and (8.7) into (8.5) including (8.4) gives (8.3) for small  $\delta$  and large k.

Write  $\nabla^{\delta} \eta = \rho^{\delta} \nabla(\rho^{-\delta} \eta)$  where  $\nabla$  is as usual the covariant derivative of the cylindrical metric on the domain and the metric induced on  $Z_{\lambda}$  from Z. Note that when  $\delta > 0$  is small, the  $L^{1,2}$  weighted norm defined using  $\nabla^{\delta}$  is uniformly (in  $\lambda$ ) equivalent to the one using  $\nabla$ . Then Proposition 8.2 implies:

Corollary 8.3 For  $\delta > 0$  small there are constants  $k_0$  and c such that for all  $\lambda$  sufficiently small and all approximate maps  $F \in \mathcal{A}_{\lambda}^{\delta_0}$  and each neck  $N_k$  with  $k \geq k_0$ , each  $\eta \in \Omega^{0,1}(F^*TZ_{\lambda})$  satisfies

$$\|\eta\|_{1,2,N_k} \le c\|D^*\eta\|_{2,N_k} + c\|\eta\|_{1,2,\partial N_k} \tag{8.9}$$

i.e.

$$\int_{N_k} \rho^{-\delta} \left( |\nabla^{\delta} W \eta|^2 + |W \eta|^2 \right) \le c \int_{N_k} \rho^{\delta} |W D^* \eta|^2 + c \int_{\partial N_k} \rho^{-\delta} \left( |\nabla^{\delta} W \eta|^2 + |W \eta|^2 \right). (8.10)$$

**Proof.** Since on each coordinate  $\partial \overline{W} = 0$  then relation (1.9) combined with condition (1.10) implies that

$$(\overline{W})^{-1}L^*(\overline{W}\eta) = L^*\eta + O(\rho|\eta|).$$

So (8.10) follows from (8.3) after replacing  $\eta$  by  $\rho^{-\delta}W\eta$  and using (7.6).  $\Box$ 

From now on we will fix  $\delta > 0$  small and generic. The following lemma can be compared to Lemma 6.6 in [RT1] and 3.10 in [LT].

**Lemma 8.4** There is a constant E > 0 such that for all  $\lambda$  sufficiently small and all approximate maps  $F \in \mathcal{A}_{\lambda}^{\delta_0}$ , the first eigenvalue of  $\mathbf{DD}^*$  is bounded below by E.

**Proof.** Suppose the claim is false. Then there are sequences  $\lambda_n, \mu_n \to 0$ , maps  $F_n : C_{\mu_n} \to Z_{\lambda_n}$  in some  $\mathcal{K}_{\delta_0}$  and (0,1) forms  $\eta_n$  along  $F_n$  with  $\mathbf{D}_n \mathbf{D}_n^* \eta_n = \varepsilon_n \eta_n$  with  $\varepsilon_n \to 0$ . In particular,

$$\varepsilon_n \int \rho^{-\delta} |W\eta_n|^2 \geq \int \rho^{-\delta} |WD_n^*\eta_n|^2 + |A\eta|^2 + |B\eta|^2. \tag{8.11}$$

where A, B as in (7.6). We may normalize the  $\eta_n$  so that the lefthand side of (8.9) is one. By the Bubble Tree Convergence Theorem there is a subsequence of the  $F_n$  that converges to a stable map  $F_0$  from  $C_0 = C_1 \cup C_2$  into  $Z_0$ , and this convergence is in  $C^{\infty}$  away from the nodes. On small compact sets K in the complement of the nodes, the  $L_s^{1,2}$  norm in the cylindrical metric is uniformly equivalent to the usual  $L^{1,2}$  norm. Standard elliptic theory implies that there is a subsequence of the  $\eta_n$  that converges in  $C^{\infty}$  on K to an  $L_s^{1,2}$  section with  $D_0^* \eta = 0$  along  $F_0 \setminus K$ . Doing this for the sequence  $K_m = \rho^{-1}([\frac{1}{m}, \infty))$  and passing to a diagonal subsequence yields a limit  $\eta$  defined on  $C_0 \setminus \{\text{nodes}\}$  with  $L_s^{1,2}$  norm at most one, and such that  $D_0^* \eta = 0$  along  $F_0$  outside the nodes. Moreover,  $\mathbf{D}_0^* \eta = 0$  weakly, i.e. for all  $\zeta \in L_s^{1,2}$ ,  $a \in T_p V$  and  $v \in T_C \mathcal{M}$ 

$$\langle \mathbf{D}_0(\zeta, a, v), \eta \rangle = 0$$

on  $C_0$ . We show this for  $a \in TV$ , the other parts being similar. On  $C_{\mu_n}$ 

$$\langle \mathbf{D}_n(a), \eta_n \rangle = \langle a, A \eta_n \rangle_{TV} \to 0$$

and  $\mathbf{D}_n(a) = D_n(\beta a) = (\overline{\partial}\beta)a + \beta \nabla_a J \circ dF_n$ . Off each neck  $N_k = \{\rho < 1/k\}$ 

$$\int_{C_u \setminus N_k} \rho^{-\delta} \langle W D_n(\beta a), \ W \eta_n \rangle \to \int_{C_0 \setminus N_k} \rho^{-\delta} \langle W D_0(\beta a), \ W \eta \rangle$$

while on  $N_k D_0(\beta a) = D_0(a) = (\nabla_a J) \circ dF_n \circ j$ . So

$$\left| \langle D_0(\beta a), \eta_n \rangle_{N_k} \right| = \left| \int_{N_k} \rho^{-\delta} \langle W D_0(\beta a), W \eta_n \rangle \right| \le \left( \int_{N_k} \rho^{-\delta} |W(\nabla_a J) \circ dF_n \circ j|^2 \right)^{1/2} \cdot \|\eta_n\|_{2,s}$$

But V is J-invariant, so  $WD_0(\beta a) = (\nabla_a J) \circ d(WF_n) \circ j + O(\rho)$  and  $|\nabla_a J|$  is bounded and  $|dWF_n| \leq \rho^{1/3}$ , so the first factor on the right hand side of the last displayed equation goes to zero as  $k \to \infty$ .

This means that  $\mathbf{D}_0^* \eta = 0$  where  $\mathbf{D}_0$  is the operator defined in (7.3). As we observed after equation (7.3), for generic  $(J, \nu)$  we have Coker  $\mathbf{D}_0 = 0$ , so  $\eta = 0$ . Therefore  $\eta_n \to 0$  in  $L^{1,2}$  on the complement of each neck  $N_k$ , which contradicts (8.9).

**Lemma 8.5** There is a constant C such that for all  $\lambda$  sufficiently small and all approximate maps  $F \in \mathcal{A}_{\lambda}^{\delta_0}$ , each  $\eta \in \Omega^{0,1}(F^*TZ_{\lambda})$  satisfies  $\|\eta\|_2 \leq C \|\mathbf{D}\mathbf{D}^*\eta\|_0$ .

**Proof.** Cover  $C_{\mu}$  by disks of radius 1 in the cylindrical metric so that each point lies in at most 10 disks. Since  $\rho$  varies by a bounded factor across each unit interval in the neck we can applying the basic elliptic estimate on each disk, multiply by  $\rho^{-\delta/2}$  and sum to get

$$\|\eta\|_{2,p,s} \le C_p \left(\|D_\mu D_\mu^* \eta\|_{p,s} + \|\eta\|_{p,s}\right)$$

for a constant  $C_p$  independent of  $\mu$ . Adding together the p=4 and p=2 inequalities we get

$$\|\eta\|_2 \le C_p \left(\|D_\mu D_\mu^* \eta\|_0 + \|\eta\|_0\right)$$

Using Lemma 8.4 and applying Holder's inequality for the weighted  $L^2$  norm

$$c \|\eta\|_{2,s}^2 \leq \|D^*\eta\|_{2,s}^2 = \langle \eta, D_\mu D_\mu^*\eta \rangle \leq \|\eta\|_{2,s} \|D_\mu D_\mu^*\eta\|_{2,s} \leq \|\eta\|_{2,s} \|D_\mu D_\mu^*\eta\|_{p,s}.$$

which combined with the previous inequality gives the desired inequality.  $\Box$ 

In the next section we will use  $P_{\mu}$  to coordinatize the normal direction to the space of approximate maps.

### 9 The Gluing Diffeomorphism

The norm (6.8) induces a topology on the space  $\operatorname{Maps}_s(C, Z_\lambda)$ . Specifically, for  $C^0$  close maps with the same label s we can write  $(f', j') = \exp_{(f,j)}(\xi, h)$  and set

$$\operatorname{dist}((C, f), (C', f')) = \|(\xi, h)\|_1 \tag{9.1}$$

This defines a topology and a distance (the inf of the lengths over all paths piecewise of the above type) on  $\operatorname{Map}_s(C, Z_{\lambda})$ . Using this distance, we will show that the moduli space of stable maps into  $Z_{\lambda}$  is close to the space of approximate maps, and that those spaces are in fact are isotopic.

We start by describing a parameterization for a neighborhood of  $\mathcal{A}^{\delta}_{\lambda}$  inside the space of maps  $(\mathcal{A}^{\delta}_{\lambda})$  is the compact set (8.1) of approximate maps). Consider the Banach space bundle  $\Lambda^{01} \to \mathcal{A}\mathcal{M}_s(\lambda)$  over  $\mathcal{A}\mathcal{M}^{\delta}_s(\lambda)$  (the model space for approximate maps) whose fiber at an approximate map  $F_{\mu}: C_{\mu} \to Z_{\lambda}$  is  $\Lambda^{0,1}(F_{\mu}^*TZ_{\lambda})$  with the norm (6.8). Write elements of  $\Lambda^{01}$  as triples  $(f, \mu, \eta)$ , with  $f \in \mathcal{M}^V_s(X) \times_{ev} \mathcal{M}^V_s(Y)$ . The map

$$\Phi_{\lambda}: \Lambda^{01}(\varepsilon) \to \operatorname{Maps}_{s}(C, Z_{\lambda} \times \mathcal{U}) \quad \text{by} \quad \Phi_{\lambda}(f, \mu, \eta) = \exp_{F_{f,\mu}, C_{\mu}}(P_{\mu}\eta)$$
(9.2)

defined on an  $\varepsilon$  neighborhood of the zero section of  $\Lambda^{01}$  agrees with the gluing map  $\Gamma_{\lambda}$  along the zero section. The following lemma shows that  $\Phi_{\lambda}$  coordinatizes a neighborhood of  $\mathcal{A}_{\lambda}^{\delta}$ .

**Proposition 9.1** There is a constant c > 0 so that for all small  $\lambda \Phi_{\lambda}$  is a diffeomorphism from an  $\varepsilon$ -neighborhood of the zero section in  $\Lambda^{01}$  onto a neighborhood of  $\mathcal{A}_{\lambda}^{\delta}$  in  $Maps_s(C, Z_{\lambda} \times \mathcal{U})$  that contains at least a  $c\varepsilon$  neighborhood of  $\mathcal{A}_{\lambda}^{\delta}$ .

**Proof.** By Lemma 3.5  $T_{F_{\mu}} \mathcal{A}_{\lambda}$  has the same dimension as Ker  $\mathbf{D}_{\mu} = (\operatorname{Im} P_{\mu})^{\perp}$ . In fact,

$$T_{F_{\mu}}\Lambda^{01} = T_{F_{\mu}}\mathcal{A}_{\lambda} \oplus \operatorname{Im} P_{\mu} \tag{9.3}$$

because any  $P_{\mu}\eta$  which lies in  $T_{F_{\mu}}\mathcal{A}_{\lambda}$  satisfies, by (8.2), Lemma 5.4 and Lemma 9.3 below,

$$||P\eta||_1 \le E ||\eta|| = E ||\mathbf{D}_{\mu}P\eta|| \le CE |\lambda|^{1/8|s|} ||P\eta||_1,$$

so, for small  $\lambda$ ,  $P\eta$  is zero.

Next fix a path  $(f_t, \mu_t)$  in  $\mathcal{AM}_s(\lambda)$  starting at  $(f_0, \mu_0)$  and let  $\xi \in T\mathcal{A}_{\lambda}$  the tangent vector at t = 0 of the corresponding path of approximate maps  $F_t = \Phi(f_t, \mu_t, 0)$ . Each element  $\tau$  in the fiber of  $\Lambda^{01}$  over  $(f_0, \mu_0)$  determines a vector field  $P\tau$  along the image of  $F_0$  in  $TZ_{\lambda}$ . After extending  $P\tau$  along  $F_t$  by parallel translation we calculate

$$d\Phi_{(f,\mu,\eta)}(\xi,h,\tau) = \frac{d}{dt} \exp_{(F_t,j_\mu)}(tP\tau) \Big|_{t=0} = \xi + P\tau.$$

Thus  $d\Phi_{\lambda}$  is an isomorphism by (9.3), so  $\Phi_{\lambda}$  is a local diffeomorphism near the zero section of  $\Lambda^{01}$ .

To show injectivity, let  $\Lambda^{01}(\varepsilon)$  be the subset of  $\Lambda^{01}$  with  $\|\eta\| \leq \varepsilon$  and suppose that injectivity fails on each  $\Lambda^{01}(\varepsilon)$ . Then for each n there exist elements  $(f_n, \mu_n, \eta_n) \neq (f'_n, \mu'_n, \eta'_n)$  in  $\Lambda^{01}(1/n)$  which have the same image under  $\Phi_{\lambda}$ . After passing to subsequences, we can assume that the  $\{(f_n, \mu_n)\}$  and  $\{(f'_n, \mu'_n)\}$  converge in the stable map topology to limits  $f: C \to Z_0$  and  $f': C' \to Z_0$  with f and f' in  $K \subset \mathcal{M}_s^V \times \mathcal{M}_s^V$  and f' and f' on the boundary of the cylindrical end compactification of  $\mathcal{M}_{g,n}$  defined at the end of Section 4. (Thus f' and f' each consist of a nodal curve together with an element of the real torus f'.)

Choose a compact region R in C which contains no nodes. Then for small  $\lambda$  we have  $F_n \to f$  and  $F'_n \to f'$  in  $C^1$  on R. Since our  $\|\cdot\|_1$  norm dominates both the  $C^0$  norm on maps and, by Lemma 4.2, the cylindrical end metric on  $\mathcal{M}_{q,n}$ ,

$$\lim_{n \to \infty} \text{dist } (C_{\mu_n}, C_{\mu'_n}) + \sup_{x \in R} \text{dist } (f(x), f'(x)) \leq \lim_{n \to \infty} (\|P\eta_n\|_1 + \|P\eta'_n\|_1)$$

$$\leq c \lim_{n \to \infty} (\|\eta_n\| + \|\eta'_n\|) = 0$$

using Lemma 8.4. Thus (i) C = C', and (ii) f and f' agree on R and therefore, as in the argument after (6.3), agree everywhere. Consequently, for large n  $(f_n, \mu_n, \eta_n)$  and  $(f'_n, \mu'_n, \eta'_n)$  lie in the region where  $\Phi_{\lambda}$  is a local diffeomorphism and are therefore equal. That establishes injectivity. The surjectivity onto an  $c\varepsilon$  neighborhood follows from the first inequality in (8.2).

The norms (6.8) for  $\lambda = 0$  induce a Banach manifold structure on  $\mathcal{M}_s^V(X) \times_{ev} \mathcal{M}_s^V(Y)$ , and hence on its cover the model space  $\mathcal{AM}_s(\lambda)$ . But the gluing map identifies  $\mathcal{AM}_s(\lambda)$  with the space of approximate maps  $\mathcal{A}_{\lambda}$ , which has a possibly different norm as a subset of the Banach space  $\mathrm{Maps}_s(C, Z_{\lambda} \times \mathcal{U})$ . The next lemma shows that these two norms on  $T\mathcal{AM}_s$  are uniformly equivalent.

**Lemma 9.2** There are constants c, C > 0, uniform on each compact  $\mathcal{K}_{\delta} \subset \mathcal{M}_{s}^{V}(X) \times_{ev} \mathcal{M}_{s}^{V}(Y)$ , so that for each tangent vector  $(\xi, h)$  to  $\mathcal{M}_{s}^{V}(X) \times_{ev} \mathcal{M}_{s}^{V}(Y)$  each of its images  $(\xi_{\mu}, h_{\mu})$  under the differential of the gluing map (6.3) satisfy

$$c\|(\xi,h)\|_1 \le \|(\xi_\mu,h_\mu)\|_1 \le C\|(\xi,h)\|_1.$$

**Proof.** Choose a path  $(f_t, j_t)$  in  $\mathcal{M}_s^V(X) \times_{ev} \mathcal{M}_s^V(Y)$  with tangent  $(\xi, h)$  at t = 0 and lift it to a path  $(F_t, \mu_t) \in \mathcal{A}\mathcal{M}_s$  with initial tangent vector  $(\xi_\mu, h_\mu)$ . By construction, the approximate maps agree with  $(f_t, j_t)$  outside the region  $A_\mu = \{\rho < 2|\mu|^{1/4}\}$  and so  $\xi_\mu = \xi$  and  $h_\mu = h$  off  $A_\mu$ . Moreover,  $\xi$  and  $\xi_\mu$  have the same average value in  $T_pV^\ell$ , so we may assume without loss of generality that this average value is 0. Then on  $A_\mu \xi_\mu = (1 - \beta_\mu)\xi$  while  $h_\mu - h$  is of order  $\dot{\mu}_t/\mu$  by Lemma 4.2. By differentiating the relation  $a_t b_t \mu_t^s = \lambda$  we see that  $\dot{\mu}_t/\mu$  is of order  $\dot{a}_t/a + \dot{b}_t/b$ . Integrating on  $A_\mu$  and using Lemma 7.1 gives

$$\|(\xi_{\mu} - \xi, h_{\mu} - h)\|_{1,A_{\mu}} \le C|\mu|^{1/8}\|(\xi, h)\|_{1}$$

uniformly on the compact K (when  $\delta < 1/2$ ).  $\square$ 

**Lemma 9.3** There is a constant C, uniform for  $f_0$  in  $\mathcal{K}_{\delta} \subset \mathcal{M}_s^V(X) \times_{ev} \mathcal{M}_s^V(Y)$ , such that for  $\lambda$  small enough the tangent vectors  $(\xi, h) \in T_F \mathcal{A}_{\lambda}$  at the approximate map  $F = F_{f,\mu}$  satisfy

$$\|\mathbf{D}_{\mu}(\xi,h)\|_{0} \leq C|\lambda|^{1/8s} (\|\xi\|_{1} + \|h\|).$$

**Proof.** This time, choose a path  $(f_t, j_t) \in \mathcal{M}_s^V \times_{ev} \mathcal{M}_s^V$  with initial tangent vector  $(\xi_0, h_0) \in \ker \mathbf{D}_f$ , lift to a path  $(F_t, \mu_t) \in \mathcal{AM}_s$ , and let  $(\xi, h)$  be the initial tangent vector to the lifted path. On the neck  $\rho < |\mu|^{1/4}$  we again have  $\xi = (1 - \beta_{\mu})\xi_0$ . The lemma follows from the pointwise estimates

$$|\mathbf{D}_{\mu}(\xi,h) - \mathbf{D}_{0}(\xi,h)| \leq |dF_{\mu} - df_{0}| \cdot (|\xi| + |h|)$$
  

$$|\mathbf{D}_{0}(\xi,h)| \leq |\mathbf{D}_{0}(\xi,h) - \mathbf{D}_{0}(\xi_{0},h_{0})| \leq |\nabla \beta_{\mu} \cdot \xi_{0}| + |\nabla \xi_{0} J \circ df_{0} \circ j| + |J \circ df_{0} \circ (h - h_{0})|$$

combined with Lemma 9.2.  $\Box$ 

**Proposition 9.4** For each  $\varepsilon > 0$ ,  $\mathcal{M}_s^{flat}(Z_\lambda)$  lies in an  $\varepsilon$ -neighborhood of  $\mathcal{A}_\lambda^{\delta}$  for all  $\lambda < \lambda_0(\varepsilon)$ .

**Proof.** As  $\lambda_n \to 0$ , any sequence  $(f_n, j_n) \in \mathcal{M}_s^{flat}(Z_{\lambda_n})$  has a subsequence which converges as in (5.1) to a limit  $f_0$  from an  $\ell$ -nodal curve  $C_0$ . Write  $(\Sigma, j_n) = (C_n, \mu_n)$  where  $C_n$  is an  $\ell$ -nodal curve close to  $C_0$  and choose  $\mu'_n = (\mu'_{n,1}, \dots, \mu'_{n,\ell})$  with  $\lambda_n = a_k b_k (\mu'_{n,k})^{s_k}$  for each k; there are |s| choices for each  $\mu'_n$  which differ by roots of unity. That data defines corresponding approximate maps  $F_n = F_{f_0, \mu'_n} : (C_0, \mu'_n) \to Z_{\lambda_n}$  in  $A_{\lambda_n}$  via (6.2). We will show that for some choice of  $\mu'_n$ 

$$\operatorname{dist}((C_n, \mu_n), (C_0, \mu'_n)) + \operatorname{dist}(f_n, F_n) < \varepsilon$$
 for large  $n$ .

Lemma 5.4 shows that  $(\mu_n/\mu_n')^s \to 1$  at each node. After passing to a subsequence and modifying our choice of  $\mu_n'$  we have  $\mu_n/\mu_n' \to 1$ . But then (4.8) shows that  $\operatorname{dist}((C_n, \mu_n), (C_0, \mu_n')) \to 0$ .

For any  $r_0 < 1/2$  the maps  $F_n$  and  $f_0$  agree on the sets  $\{\rho \geq r_0\}$  for all large n. Inside the region  $A(r_0)$  near each node where  $\rho \leq r_0$ , the vector  $\xi_n = F_n - f_n = \beta_\mu (f_0 - f_n)$  can be written as  $\xi_n = (\zeta_n, \bar{\xi}_n)$  as in (6.5). But  $\bar{\xi} \to 0$  because  $f_n \to f_0$  in  $C^0$ , and Lemma 5.5 implies that  $\|\zeta\|_{1,A(r_0)} \leq cr_0^{1/6}$ . Taking  $r_0$  small enough and using the fact that outside the neck  $A(r_0)$  we have uniform convergence implies  $\operatorname{dist}(f_n, F_n) = c(\|\zeta\|_1 + |\bar{\xi}|) < \varepsilon$  for large n.

The next step is to correct each approximate map  $F_{f,\mu} \in \mathcal{A}_{\lambda}$  to get  $(F'_{\mu}, j'_{\mu})$  a true  $(J, \nu)$ -holomorphic map. More precisely,  $(F'_{\mu}, j'_{\mu})$  will be a solution of the equation

$$\overline{\partial}_j f = \nu_f \quad \text{where} \quad (f, j) = \exp_{F_\mu, j_\mu}(P_\mu \eta)$$
(9.4)

and  $\eta \in \Lambda^{0,1}(F_{\mu}^*TZ_{\lambda})$ .

**Proposition 9.5** There are constants  $\varepsilon$ ,  $\lambda_0$  and C (uniform on  $\mathcal{K}_{\delta} \subset \mathcal{M}_s^V(X) \times_{ev} \mathcal{M}_s^V(Y)$ ) such that for each  $f \in \mathcal{K}_{\delta}$  and  $0 < |\lambda| < \lambda_0$  equation (9.4) has a unique solution  $\eta \in \Lambda^{0,1}(F_{\mu}^*TZ_{\lambda})$  in the ball  $\|\eta\| \le \varepsilon$ , and that solution is smooth and satisfies  $\|\eta\| \le C|\lambda|^{\frac{1}{8s}}$ .

**Proof.** If we write  $(f,j) = \exp_{F_{\mu},j_{\mu}}(\zeta)$  where  $\zeta = (\xi,h)$  then

$$\overline{\partial}_{j}f - \nu_{f} = \overline{\partial}_{j_{\mu}}F_{\mu} - \nu_{F_{\mu}} + \mathbf{D}_{\mu}(\zeta) + Q_{\mu}(\zeta)$$
(9.5)

where  $\mathbf{D}_{\mu}$  is the linearization at  $(F_{\mu}, j_{\mu})$  and the quadratic  $Q_{\mu}$  satisfies (c.f. [F])

$$||Q_{\mu}(\zeta_1) - Q_{\mu}(\zeta_2)|| \leq C (||\zeta_1||_1 + ||\zeta_2||_1)||\zeta_1 - \zeta_2||_1$$
(9.6)

Taking  $\zeta = P_{\mu}\eta$  and noting that  $D_{\mu}P_{\mu}\eta = \eta$ , equation (9.4) becomes

$$\eta + Q(P_{\mu}\eta) = v \quad \text{where} \quad v = \nu_{\mu} - \overline{\partial}_{\mu}f_{\mu}$$
(9.7)

Define an operator  $T_{\mu}$  on the Banach space obtained by completing  $\Omega^{0,1}(f_{\mu}^*TX)$  in our norm (6.9) by

$$T_{\mu}\eta = v - Q_{\mu}(P_{\mu}\eta).$$

Using (9.6) and (8.2)

$$||T_{\mu}\eta_{1} - T_{\mu}\eta_{2}|| \leq C (||P_{\mu}\eta_{1}||_{1} + ||P_{\mu}\eta_{2}||_{1})||P_{\mu}(\eta_{1} - \eta_{2})||_{1}$$
  
$$\leq C E^{2} (||\eta_{1}|| + ||\eta_{2}||) \cdot ||\eta_{1} - \eta_{2}||.$$

Choosing  $\varepsilon < 1/(4CE^2)$ , when  $||T_{\mu}(0)|| \le \varepsilon/2$  then  $T_{\mu} : B(0,\varepsilon) \to B(0,\varepsilon)$  is a contraction on the ball of radius  $\varepsilon$ . Therefore  $T_{\mu}$  has a unique fixed point  $\eta$  in the ball, and  $||\eta|| \le 2||T_{\mu}(0)||$ . Finally, since  $\eta \in L^4_{loc}$  we have  $\zeta = P\eta \in L^{1,4}_{loc}$  with  $D\zeta + Q(\zeta) = Pv \in C^{\infty}$ . Elliptic regularity then shows that  $\zeta$  and  $\eta$  are smooth.  $\square$ 

# 10 Convolutions and the Sum Formula for Flat Maps

We can now assemble the analysis of the previous several sections to show that the approximate moduli space, which is built from maps into  $Z_0$ , is a good model of the moduli space of stable maps into the symplectic sum  $Z_{\lambda}$ . Recall that in Sections 3 and 5 we showed that as  $\lambda \to 0$  stable maps into  $Z_{\lambda}$  limited to maps into  $Z_0$  and that the complex structure  $\mu$  on their domains are determined by the limit map up to a finite ambiguity corresponding to the different solutions of the equation  $ab\mu^s = \lambda$ . That led to the definition of the model moduli space  $\mathcal{AM}_s$  in Section 6. On the other hand, each element of  $\mathcal{AM}_s$  defines an approximate holomorphic map by equation (6.2); for each  $\lambda$  this gives the gluing map

$$\Gamma_{\lambda} : \mathcal{AM}_s \stackrel{\approx}{\longrightarrow} \mathcal{A}_s(\lambda) \subset \operatorname{Maps}_s(C, Z_{\lambda} \times \mathcal{U})$$
 (10.1)

whose image  $A_s(\lambda)$  we call the space of approximate maps. And indeed, Proposition 9.5 shows that each such approximate map can be uniquely perturbed to be true  $(J, \nu)$ -holomorphic map.

In this section we will show that  $\mathcal{A}_s(\lambda)$  is isotopic to  $\mathcal{M}_s(Z_\lambda)$  through an isotopy compatible with the evaluation maps. Thus  $\mathcal{A}\mathcal{M}_s(\lambda)$  keeps track of the fundamental homology class  $[\mathcal{M}_s(Z_\lambda)]$  which defines the GW and TW invariants of  $Z_\lambda$  (we continue to assume that all maps have been stabilized as in Remark 3.4). Passing to homology, we then define a "convolution" operation and establish a formula of the form

$$TW_X^V * TW_Y^V = TW_Z (10.2)$$

under the assumption that all curves contributing to the invariants are V-flat (this condition will be eliminated in Section 12).

We noted in (3.11) that as  $\lambda \to 0$  the limits of the  $\delta$ -flat maps into  $Z_{\lambda}$  lie in the compact set  $\mathcal{K}_{\delta}$  of  $\mathcal{M}^{V}(X) \times_{ev} \mathcal{M}^{V}(Y)$ . We will work on the corresponding compact sets  $\mathcal{AM}_{s}^{\delta}$  and  $\mathcal{A}_{s}^{\delta}(\lambda)$  defined in (8.1).

**Theorem 10.1** Fix an ordered sequence s and write  $|s| = \prod s_i$ . For generic  $(J, \nu)$  and small  $|\lambda|$ , there is an |s|-fold cover  $\mathcal{AM}_s^{\delta}$  of  $\mathcal{K}_{\delta}$  and a diagram

where the top arrow is a diffeomorphism onto its image and is isotopic to the restriction of (10.1) to  $\mathcal{AM}_s^{\delta}$ . The diagram commutes up to homotopy. Furthermore, there is a constant  $c = c(\delta)$  so that the image of  $\Phi_{\lambda}^1$  consists of maps which are  $(\delta - c\lambda)$ -flat, and the image contains all  $(\delta + c\lambda)$ -flat maps in  $\mathcal{M}_s(Z_{\lambda})$ .

**Proof.** For each  $(f_0, \mu) \in \mathcal{AM}_s$  the gluing map  $\Gamma_{\lambda}$  associates a smooth curve  $C_{\mu}$  and an approximate map  $F_{\mu}: C_{\mu} \to Z_{\lambda}$ . By Proposition 9.1 any pair  $(f', C_{\mu'})$  that is  $L^1_s$  close to  $\Gamma_{\lambda}(f, \mu)$  can be uniquely written as

$$\Phi_{\lambda}(f,\mu,\eta) = \exp_{F_{f,\mu},C_{\mu}}(P_{\mu}\eta) \tag{10.4}$$

for some  $L_s^0$  section  $\eta$  of the bundle  $\Lambda^{0,1}$  with  $\|\eta\| < \varepsilon$ . Proposition 9.5 then used a fixed point theorem to show that for small  $|\lambda|$  there is a unique such  $\eta = \eta(f, \mu)$  such that (10.4) is  $(J, \nu)$ -holomorphic. Then

$$\Phi_{\lambda}^{t}(f,\mu,\eta) = \exp_{F_{t,\mu},C_{\mu}} (tP_{\mu}\eta(f,\mu))$$

is a smooth 1-parameter family of maps from  $\mathcal{AM}_s^{\delta}$  to  $\mathrm{Maps}_s(C, Z_{\lambda} \times \mathcal{U})$  with  $\Phi_{\lambda}^0 = \Gamma_{\lambda}$  and the image of  $\Phi_{\lambda}^1$  lying in the  $(\delta - c\lambda)$ -flat maps in  $\mathcal{M}_s(Z_{\lambda})$ . The uniqueness of  $\eta$  in the fibers of  $\Lambda^{0,1}$ , combined with Proposition 9.1 implies that the  $\Phi_{\lambda}^1$  is injective.

It remains to show that  $\Phi_{\lambda}^1$  is surjective. But Proposition 9.1 shows that (10.4) is onto at least a  $c\varepsilon$  neighborhood of  $\mathcal{A}_{\lambda}^{2\delta}$  and Proposition 9.4 implies that  $\mathcal{M}_{s}^{flat}(Z_{\lambda})$  lies in that neighborhood when  $|\lambda|$  is small enough. Hence for  $|\lambda|$  small, each element of  $\mathcal{M}_{s}^{flat}(Z_{\lambda})$  can be written in the form (10.4) with  $(F, C_{\mu}) \in \mathcal{A}_{\lambda}^{\delta+c\lambda}$  and  $\|\eta\|_{0} \leq \varepsilon$ ; this  $\eta$  must then be the unique fixed point  $\eta(f, \mu)$  of Proposition 9.5. Thus  $\Phi_{\lambda}^{1}$  is surjective.  $\square$ 

Diagram 10.3 leads to our first formula expressing the absolute invariants of a symplectic sum  $Z=Z_{\lambda}$  in terms of the relative invariants of X and Y. Recall that the relative invariant  $GW_X^V$  is obtained by forming the space  $\overline{\mathcal{M}}_{\chi,n,s}^V(X,A)$  of relatively stable maps and pushing forward its fundamental homology class by the map

$$\varepsilon_V : \overline{\mathcal{M}}_{\chi,n,s}^V(X,A) \to \widetilde{\mathcal{M}}_{\chi,n} \times X^n \times \mathcal{H}_{X,A,s}^V.$$
 (10.5)

We can also consider the space of stable maps from compact, not necessarily connected domains by taking the union of products of  $\overline{\mathcal{M}}_{\chi,n,s}^V(X,A)$  and again pushing forward in homology. The resulting class in the homology of  $\widetilde{\mathcal{M}}_{\chi,n} \times X^n \times \mathcal{H}_{X,A,s}^V$  is the relative TW invariant (1.18). As we observed in the introduction (see Figure 1), it is the TW invariant that will appear in the symplectic sum formula.

To proceed, then we should replace the vertical arrows in Diagram 10.3 by the above maps  $\varepsilon_V$  and pass to homology. We will do that in two steps, first incorporating the spaces  $\mathcal{H}_X^V$  and then including the  $X^n$ . In each case we will see that the operation of gluing maps defines an extension of the bottom arrow in Diagram 10.3, which we examine in homology.

The Convolution Operation We can glue a map  $f_1$  into X to a map  $f_2$  into Y provided the images meet V at the same points with the same multiplicity. The domains of  $f_1$  and  $f_2$  glue according to the attaching map  $\xi$  of (3.8), while the images determine elements of the intersection-homology spaces  $\mathcal{H}_{X,A,s}^V$  and  $\mathcal{H}_{Y,A,s}^V$  which glue according to the map g of (3.10). The convolution operation records the effect of these gluings at the level of homology.

For each s the attaching map (3.8) defines a bilinear form

$$(\xi_{\ell})_*: H_*(\widetilde{M}; \mathbb{Q}) \otimes H_*(\widetilde{M}; \mathbb{Q}) \longrightarrow H_*(\widetilde{M}; \mathbb{Q})$$

for  $\ell = \ell(s)$ . Similarly, for each s the map g from (3.10) induces a bilinear form on the homology of  $\mathcal{H}_Y^V \times \mathcal{H}_Y^V$  with values in  $RH_2(Z)$ , the (rational) group ring of  $H_2(Z)$ , namely

$$\langle , \rangle : H_*(\mathcal{H}_X^V; \mathbb{Q}) \otimes H_*(\mathcal{H}_Y^V; \mathbb{Q}) \longrightarrow RH_2(Z)$$

$$\langle h, h' \rangle_s = g_* \left[ h \times h' |_{\varepsilon^{-1}(\Delta_s)} \right] = \sum_{A \in H_2(Z)} g_* [\Delta_{A,s} \cap (h \times h')] t_A.$$

This last equality holds because  $\varepsilon^{-1}(\Delta_s)$  is the union of components  $\Delta_{A,s} = \varepsilon^{-1}(\Delta_s) \cap g^{-1}(A)$ .

Combining the two bilinear forms gives the convolution operator that describes how homology classes of maps combine in the gluing operation.

**Definition 10.2** The convolution operator

\*: 
$$H_*(\widetilde{\mathcal{M}} \times \mathcal{H}_X^V; \mathbb{Q}[\lambda]) \otimes H_*(\widetilde{\mathcal{M}} \times \mathcal{H}_Y^V; \mathbb{Q}[\lambda]) \longrightarrow H_*(\widetilde{\mathcal{M}}; RH_2(Z)[\lambda])$$

is given by

$$(\kappa \otimes h) * (\kappa' \otimes h') = \sum_{s} \frac{|s|}{\ell(s)!} \lambda^{2\ell(s)} \left( \xi_{\ell(s)} \right)_* (\kappa \otimes \kappa') \langle h, h' \rangle_s$$
 (10.6)

The right hand side of (10.6) includes three numerical factors which keep track of how maps glue when we form the symplectic sum. Recall that the powers of  $\lambda$  record the euler characteristic in the generating series of the invariants (1.3) and (1.17); the factor  $\lambda^{2\ell(s)}$  in (10.6) the reflects the relation (3.7) between the euler characteristics when we glue along  $\ell(s)$  points. The factor |s| is the degree of the covering in Theorem 10.1; this reflects the fact that each stable map into  $Z_0$  can be smoothed in  $|s| = s_1 \cdots s_\ell$  ways. Finally, note that elements in the space  $\mathcal{M}_s^{flat}(Z_\lambda)$  in Diagram 10.3 are labeled maps, i.e. they have  $\ell(s)$  numbered curves on their domains as explained at the end of section 3. But the GW and TW invariants of  $Z_\lambda$  are defined using the space of unlabeled stable maps, which is the quotient of the space of labeled maps by the action of the symmetric group. That accounts for the factor in  $1/\ell(s)$ ! is (10.6).

Since  $\mathcal{H}_X^V$  is the disjoint union of components  $\mathcal{H}_{X,A,s}^V$  with  $A \in H_2(X)$  and  $\deg s = A \cdot V$ , there is an isomorphism

$$H_*(\mathcal{H}_X^V) \cong \sum_{A \text{ deg } s=A \cdot V} H_*(\mathcal{H}_{X,A,s}^V) t_A.$$

Below, we will identify  $h \in H_*(\mathcal{H}_X^V)$  with  $\sum_A h_A t_A$ , where  $h_A$  are its components in  $H_*(\mathcal{H}_{X,A}^V)$ .

**Example 10.3** The formula for the convolution simplifies when there are no rim tori in X and Y, and therefore in Z (c.f. (1.13)). Then (i) the relative invariants have an expansion of the form (A.3), (ii) the map g of (3.10) is the restriction to the diagonal  $\Delta_s \subset V^s \times V^s$ , and (iii) the h part of the convolution (10.6) is then given by the cap product with the Poincaré dual of the diagonal:

$$g_* \left[ h \times h' \big|_{\Delta_{\mathbf{s}}} \right] = \operatorname{PD}(\Delta_{\mathbf{s}}) \cap (h \times h').$$

We can then 'split the diagonal' by fixing a basis  $\{C^p\}$  of  $H^*(\bigsqcup_s V^s)$  and writing

$$\operatorname{PD}(\Delta_{\mathbf{s}}) = \sum_{p,q} Q_{p,q}^{V} C^{p} \times C^{q} = \sum_{p} C^{p} \times C_{p}$$

where  $Q_{p,q}^V$  is the intersection form of  $V^s$  for the basis  $\{C^p\}$  and  $C_p = \sum Q_{p,q}^V C^q$  is the dual basis. If  $\{\gamma^i\}$  is a basis of  $H_*(V)$ , let  $\{\mathbf{C}_m\}$  be the basis (A.4) of  $H^*(\bigsqcup_s V^s)$  corresponding to  $\{\gamma^i\}$  and let  $\{\mathbf{C}_{m^*}\}$  be the one corresponding to the dual basis  $\{\gamma_i\}$  (with respect to  $Q^V$ ). The convolution then has the more explicit form

$$(\kappa \otimes h) * (\kappa' \otimes h') = \sum_{m} \frac{|m|}{m!} \lambda^{2\ell(m)} \left( \xi_{\ell(m)} \right)_* (\kappa \otimes \kappa') \mathbf{C}_m^*(h) \mathbf{C}_{m^*}^*(h'). \tag{10.7}$$

In passing from s to m, we used the fact that each fixed sequence m corresponds to  $\binom{\ell(s)}{(m_{a,i})} = \frac{\ell(s)!}{m!}$  ordered sequences s.

More generally, let X be a symplectic manifold with two disjoint symplectic submanifolds U and V with real codimension two. Suppose that V is symplectically identified with a submanifold of similar triple (Y, V, W) and that the normal bundles of  $V \subset X$  and  $V \subset Y$  have opposite chern classes. Let (Z, U, W) be the resulting symplectic sum. In this case, (3.10) is replaced by

$$g: \mathcal{H}_X^{U,V} \times_{\varepsilon} \mathcal{H}_Y^{V,W} \to \mathcal{H}_Z^{U,W}$$
 (10.8)

which combines with the map  $\xi_{\ell(s)}$  to give the convolution operator

\*: 
$$H_*(\widetilde{\mathcal{M}} \times \mathcal{H}_X^{U,V}; \mathbb{Q}[\lambda]) \otimes H_*(\widetilde{\mathcal{M}} \times \mathcal{H}_Y^{V,W}; \mathbb{Q}[\lambda]) \longrightarrow H_*(\widetilde{\mathcal{M}} \times \mathcal{H}_Z^{U,W}; \mathbb{Q}[\lambda])$$
 (10.9)

as in (10.6). It describes how homology classes of maps combine in the gluing operation for the symplectic sum.

Finally, we include the evaluation maps which record the images of the n marked points. These combine with the projections from (2.7) to give the diagram

which commutes up to homotopy. We can also include the spaces  $\widetilde{\mathcal{M}}$  of curves from Diagram 10.3. Pushing forward then gives  $\pi_{0*}(TW_X^V*TW_Y^V) = \pi_{\lambda*}(TW(Z_{\lambda}))$ .

**Theorem 10.4** Assume that all curves contributing to the invariants are flat along V. Then (10.2) holds in the sense that for any  $\alpha_0 \in \mathbb{T}(H^*(Z_0))$ 

$$TW_Z^{U \cup W}(\pi^* \alpha_0) = \left( TW_X^{U \cup V} * TW_Y^{V \cup W} \right) (\pi_0^* \alpha_0). \tag{10.11}$$

**Proof.** It suffices to verify this for decomposable elements  $\alpha_0 = \alpha_0^1 \otimes \cdots \otimes \alpha_0^n$ . Let  $\alpha_V^k$ ,  $\alpha_X^k$ ,  $\alpha_Y^k$  denote the restriction of  $\alpha_0^k$  to V, X and respectively Y. We can then choose geometric representatives  $B_V^k$  of the Poincaré dual of  $\alpha_V^k$  in V and Poincaré duals  $B_X^k$  of  $\alpha_X^k$  in X and  $B_Y^k$  of  $\alpha_Y^k$  in Y which intersect V transversely such that moreover  $B_X^k \cap V = B_Y^k \cap V = B_V^k$ . Then the inverse image under  $\pi_\lambda$  of  $B_X^k \cup B_Y^k$  gives a continuous family of geometric representatives

 $B_{\lambda}^{k}$  of the Poincaré dual of  $\pi^{*}\alpha_{0}^{k}$  in  $H^{*}(Z_{\lambda})$ . The theorem then follows from Theorem 10.1 by cutting down the moduli spaces on the left of Diagram 10.10 by  $(B_{X}, B_{Y})$  and the ones on the

right by  $B_{\lambda}$ . Constraints in  $H^*(\widetilde{\mathcal{M}})$  are handled similarly. The details of such arguments are standard (c.f. [RT1]).  $\square$ 

We should comment on how the assumption that all maps are  $\delta$ -flat enters the above proof. Notice that in the statement of Theorem 10.1 the  $\delta$ -flat maps in  $\mathcal{AM}_s$  are paired with maps in  $\mathcal{M}_s(Z_\lambda)$  which are not exactly  $\delta$ -flat — there is a slight variation in  $\delta$ . But when all contributing maps are flat, the cut-down moduli space  $\operatorname{ev}^{-1}(B_\lambda) \subset \overline{\mathcal{M}}(Z_\lambda)$  limits as  $\lambda \to 0$  to a compact subset of the open set  $\mathcal{M}_s \times_{ev} \mathcal{M}_s$  as in (3.11). Hence for sufficiently small  $\delta$  the set of elements of the limit set which are  $\delta$ -flat is the same as the set of  $2\delta$ -flat elements, so the variation in  $\delta$  is inconsequential.

Theorem 10.4 is a formula for the TW invariants evaluated on only certain constraints in  $H^*(Z_{\lambda})$  — those of the form  $\pi^*(\alpha_0)$ . The following definition characterizes those constraints. It is based on the diagram induced by the collapsing maps of (2.7)

$$\mathbb{T}(H^*(Z_0))$$

$$\pi^* \swarrow \qquad \searrow \pi_0^* \qquad (10.12)$$

$$\mathbb{T}(H^*(Z)) \qquad \qquad \mathbb{T}(H^*(X) \oplus H^*(Y))$$

**Definition 10.5** We say that a constraint  $\alpha \in \mathbb{T}(H^*(Z))$  separates as  $(\alpha_X, \alpha_Y)$  if there exists an  $\alpha_0 \in \mathbb{T}(H^*(Z_0))$  so that  $\pi^*\alpha_0 = \alpha$  and  $\pi_0^*(\alpha_0) = (\alpha_X, \alpha_Y) \in \mathbb{T}(H^*(X) \oplus H^*(Y))$ .

Here are three observations to help clarify which classes  $\alpha \in H^*(Z)$  separate. These follow by combining the Mayer-Vietoris sequences for  $Z_{\lambda} = (X \setminus V) \cup (Y \setminus V)$ 

$$H^{*-1}(S_V) \xrightarrow{\delta^*} H^*(Z) \xrightarrow{i^*} H^*(X \setminus V) \oplus H^*(Y \setminus V) \longrightarrow H^*(S_V) \xrightarrow{\delta^*}$$

and the similar one for  $Z_0$  with the Gysin sequence for  $p: S_V \to V$ 

$$H^{*-2}(V) \xrightarrow{\cup c_1} H^*(V) \xrightarrow{p^*} H^*(S_V) \xrightarrow{p_*} H^{*-1}(V).$$
 (10.13)

- (a) When the first map in (10.13) is injective then all classes  $\alpha$  separate. In dimension four, that occurs whenever the normal bundle of V in X is topologically non-trivial.
- (b) In general the separating classes are those  $\alpha$  for which  $j^*(\alpha) \in H^*(S_V)$  is in the image of the second map in (10.13).
- (c) the decomposition  $(\alpha_X, \alpha_Y)$ , if it exists, is unique only up to elements in the image of  $\delta_X^* \oplus \delta_Y^* : H^{*-1}(S_V) \to H^*(X) \oplus H^*(Y)$  (the elements that can be "pushed to either side").

Using Definition 10.5 and for simplicity taking U and W to be empty, Theorem 10.4 becomes:

**Theorem 10.6** Suppose that all curves contributing to the invariants are flat along V and  $\alpha$  separates as  $(\alpha_X, \alpha_Y)$ . Then

$$TW_Z(\alpha) = \left(TW_X^V * TW_Y^V\right)(\alpha_X, \alpha_Y). \tag{10.14}$$

Note that when  $(\alpha_X, \alpha_Y)$  decomposes as  $\alpha = \alpha_X \otimes \alpha_Y$  the right hand side is  $TW_X^V(\alpha_X) * TW_Y^V(\alpha_Y)$ , but in general  $(\alpha_X, \alpha_Y)$  is a sum of tensors of the form  $(\alpha_X^1 + \alpha_Y^1) \otimes \cdots \otimes (\alpha_X^k + \alpha_Y^k)$  and the right hand side of (10.14) is the corresponding sum.

To focus on the decomposable case we make another definition: we say  $\alpha$  is supported off the neck if the restriction  $j^*(\alpha) \in H^*(S_V)$  vanishes. In that case  $\alpha$  separates into relative classes  $\alpha_X \in H^*(X,V)$  and  $\alpha_Y \in H^*(Y,V)$ , generally in several ways. For each such decomposition Theorem 10.6 gives

$$TW_Z(\alpha_X, \alpha_Y) = TW_X^V(\alpha_X) * TW_Y^V(\alpha_Y). \tag{10.15}$$

This was the formula described in [IP3].

**Example 10.7** Take  $\alpha$  to be the Poincare dual of a point in Z. This constraint is supported off the neck and has two independent decompositions depending whether the point is in X or Y.

**Example 10.8** Suppose  $\alpha = \alpha_X \otimes \alpha_Y$  is supported off the neck and there are no rim tori in (X, V) and (Y, V) and that all curves contributing to the invariants are V-flat. Then we can choose a basis of  $H^*(V)$  and expand the relative TW invariants as in Example 10.3. Combining (10.15) with (10.7) gives the explicit formula

$$TW_{\chi,A,Z}(\alpha_X,\alpha_Y) = \sum_{\substack{A=A_1+A_2\\\chi_1+\chi_2-2\ell(m)=\chi}} \sum_{m} \lambda^{2\ell(m)} \frac{|m|}{m!} TW_{\chi_1,A_1,X}^V(\alpha_X;C_m) \cdot TW_{\chi_2,A_2,Y}^V(C_{m^*};\alpha_Y).$$

Note that from the definition of relative invariants, the only terms contributing are those for which  $A_1 \cdot V = \ell(m) = A_2 \cdot V$ . E. Getzler has pointed out that the formula above can be neatly expressed in terms of the generating series (A.6) and the intersection matrix  $Q^V$  of V, specifically

$$TW_Z(\alpha_X,\alpha_Y) = \exp\left(\sum_{a,i,j} a\lambda^2 \ Q_{ij}^V \ \frac{\partial}{\partial z_{a,i}} \ \frac{\partial}{\partial w_{a,j}}\right) \left(TW_X^V(\alpha_X)(z) \cdot TW_Y^V(\alpha_Y)(w)\right) \bigg|_{z=w=0}.$$

Because the decomposition of separating constraints  $\alpha$  is not unique, we can often choose several different decompositions, and use Theorem 10.14 to get several expressions for the same TW invariant. That yields relations among relative TW invariants. In Section 15 we will use that idea to derive recursive formulas which determine the relative invariants in some interesting cases.

# 11 The space $\mathbb{F}$ and the S-matrix

Starting from the normal bundle  $N_X V$  of V in X, we can form the  $\mathbb{P}^1$  bundle

$$\mathbb{F} = \mathbb{F}_V = \mathbb{P}(N_X V \oplus \mathbb{C})$$

over V by projectivizing the sum of the normal bundle  $N_XV$  and the trivial complex line bundle. Let  $\pi: \mathbb{F} \to V$  be the projection map. In  $\mathbb{F}$ , the zero section  $V_0$  and the infinity section  $V_{\infty}$  are disjoint symplectic submanifolds, both symplectomorphic to V. Moreover, note that  $\mathbb{F}\#_V\mathbb{F} = \mathbb{F}$ .

Under the natural identification of  $V_0$  with  $V_{\infty}$ , the convolution operation (10.9) defines an algebra structure on  $H_*(\widetilde{\mathcal{M}} \times \mathcal{H}_{\mathbb{F}}^{V,V}; \mathbb{Q}[\lambda])$ . That allows us to multiply by TW invariants. Of particular interest are the invariants with no constraints on the image, that is  $TW_{\mathbb{F}}^{V,V}(\alpha)$  with  $\alpha = 1$ , which give an operator

$$\left[ TW_{\mathbb{F}}^{V,V}(1) \right] * : H_{*}(\overline{\mathcal{M}} \times \mathcal{H}_{\mathbb{F}}^{V}; \mathbb{Q}[\lambda]) \to H_{*}(\overline{\mathcal{M}} \times \mathcal{H}_{\mathbb{F}}^{V}; \mathbb{Q}[\lambda])$$
(11.1)

defined by a power series as in (1.17). This operator is key to the general symplectic sum formula given in the next section. In this section we describe (11.1) and its inverse and develop some examples.

Each  $(J, \nu)$ -holomorphic bubble map f into  $\mathbb{F}$  projects to a map  $f_V = \pi \circ f$  into V. Although  $f_V$  may not be  $(J, \nu)$ -holomorphic, we can still ask whether  $f_V$  is stable, using the second definition of stability given after (1.1), namely f is stable if its restriction to each unstable domain component is non-trivial in homology.

**Definition 11.1** A  $(V_0, V_\infty)$ -stable map  $f: C \to \mathbb{F}$  is  $\mathbb{F}$ -trivial if each of its components is an unstable rational curve whose image represents a multiple of the fiber F of  $\mathbb{F}$ .

Thus the  $\mathbb{F}$ -trivial curves are rational curves representing dF with one marked point on the zero section and one on the infinity section, both intersecting with multiplicity d. Let  $\mathcal{M}_{\mathbb{I}}$  denote the set of  $\mathbb{F}$ -trivial maps in  $\mathcal{M}_{\mathbb{F}}^{V_0,V_{\infty}}$  and consider the disjoint union

$$\mathcal{M}_{\mathbb{F}}^{V_0,V_{\infty}} = \mathcal{M}_{\mathbb{I}} \cup \mathcal{M}_{R} \tag{11.2}$$

where  $M_R$  is the set of non- $\mathbb{F}$ -trivial maps.

For the next lemma we fix a metric g' on  $\mathbb{F}$  for which  $\pi: \mathbb{F} \to V$  is a Riemannian submersion. The procedure described in the appendix of [IP4] then constructs a compatible triple  $(\omega, J, g)$  on  $\mathbb{F}$  for which  $\pi$  is holomorphic and is a Riemannian submersion. Using this metric, each perturbation term  $\nu_V$  on V has a horizontal lift  $\pi^*\nu_V$  in  $\Omega^{0,1}(T\mathbb{F})$ . We will call such a structure  $(\omega, J, g, \pi^*\nu_V)$  a submersive structure. For submersive structures, each  $(J, \pi^*\nu_V)$ -holomorphic map (f, j) into  $\mathbb{F}$  projects to a  $(J, \pi^*\nu_V)$ -holomorphic map  $(\pi \circ f, j)$  into V.

**Lemma 11.2** (a)  $\mathcal{M}_{\mathbb{I}}$  is both open and closed. The corresponding decomposition of (11.1) is

$$TW_{\mathbb{F}}^{V,V}(1) = \mathbb{I} + R^{V,V} \tag{11.3}$$

that is, the  $\mathbb{F}$ -trivial maps contribute the identity to the TW invariant.

- (b) The non-F-trivial maps have  $E(f_V) \geq \alpha_V$ , where  $\alpha_V$  is the constant of Definition 3.1.
- (c) For each fixed A, n and  $\chi$ , the corresponding term in the convolution  $R^m = R * \cdots * R$  vanishes for m large enough. Therefore, the inverse of TW is well defined by:

$$\left(TW_{\mathbb{F}}^{V,V}(1)\right)^{-1} = \sum_{m=0}^{\infty} (-1)^m R^m. \tag{11.4}$$

- **Proof.** (a) Clearly  $\mathcal{M}_{\mathbb{I}}$  is closed. To show that the complement of  $\mathcal{M}_{\mathbb{I}}$  is closed, suppose that a sequence  $(f_i)$  in the complement converges to a trivial map f in the topology of the space of stable maps. Then the homology classes converge so, after passing to a subsequence, we can assume that each  $f_i$  represents dF. Similarly, the stabilizations of the domains converge in the Deligne-Mumford space, so we can assume that all domain components of each  $f_i$  are unstable. But then the  $f_i$  lie in  $\mathcal{M}_{\mathbb{I}}$ . We conclude that  $\mathcal{M}_{\mathbb{I}}$  is both open and closed. Finally, the decomposition (11.2) gives splitting (11.3) of the TW invariant because convolution by elements of  $\mathcal{M}_{\mathbb{I}}$  is the identity.
- (b) If  $E(f_V) < \alpha_V$  then, as in the proof of Lemma 1.5 of [IP4], every component of the domain is unstable and  $f_V$  is trivial in homology and therefore f represents a multiple of F.
- (c) For each  $(J, \nu)$ , we shall bound the number N for which there are maps in the moduli space defining the convolution  $\mathbb{R}^N$ . That moduli space consists of maps f from a domain C (whose Euler class  $\chi$  and number n of marked points is fixed) to the singular manifold  $\mathbb{F}\#\cdots\#\mathbb{F}$  obtained from N copies of  $\mathbb{F}$  by identifying the infinity section of one with the zero section of the next. Furthermore, these f decompose as  $f = \bigcup f^j$  where  $f^j$  is a map from some of the components of C into the  $j^{\text{th}}$  copy of  $\mathbb{F}$ .

Fixing such an f, let  $N_1$  be the number of  $f^j$  whose domain has at least one stable component  $C_j$ . These components appear in the stabilization st(C). But st(C) lies in the space  $\mathcal{M}_{\chi,n}$  of stable curves, and hence has at most dim  $\mathcal{M}_{\chi,n}$  components. This gives an explicit bound for  $N_1$  in terms of  $\chi$  and n.

The remaining  $N_2 = N - N_1$  of the  $f^j$  each have a domain component with  $\pi_*[f^j(C_j)] \in H_2(V)$  non-trivial, so satisfy  $E(\pi \circ f^j) > \alpha_V$  by (b) above. We therefore have

$$N_2 \alpha_V \leq \sum E(\pi \circ f^j) \leq E(\pi \circ f) \leq C[A(\pi(f)) + C_{\nu}],$$

where the first sum is over those j contributing to  $N_2$  and the last inequality is as in the proof of Lemma 12.1. Since the symplectic area  $A(\pi(f))$  of the projection is a topological quantity, this bounds  $N_2$  and hence N.  $\square$ 

**Definition 11.3** The S-matrix is defined to be the inverse of the TW invariant of Lemma 11.2:

$$S_V = \left(TW_{\mathbb{F}}^{V,V}(1)\right)^{-1}.$$

(Note that this depends not just on V but on  $N_V$  and the 1-jet of  $(J, \nu)$  along V.)

The symplectic sum of (X, U, V) and  $(\mathbb{F}, V_{\infty}, V_0)$  along  $V = V_{\infty}$  is a symplectic deformation of (X, U, V), so has the same TW invariant. The convolution then defines a operation

$$H_*(\widetilde{\mathcal{M}} \times \mathcal{H}_X^{U,V}; \mathbb{Q}[\lambda]) \otimes H_*(\widetilde{\mathcal{M}} \times \mathcal{H}_{\mathbb{F}}^{V,V}; \mathbb{Q}[\lambda]) \longrightarrow H_*(\widetilde{\mathcal{M}} \times \mathcal{H}_X^{U,V}; \mathbb{Q}[\lambda]).$$

Thus for each choice of constraints  $\alpha \in \mathbb{T}(\mathbb{F}, V_{\infty} \cup V_0)$ , the TW invariant of  $\mathbb{F}$  relative to its zero and infinity section defines an endomorphism

$$TW_{\mathbb{F}}^{V_{\infty},V_0}(\alpha) \in \text{End } \left(H_*(\widetilde{\mathcal{M}} \times \mathcal{H}_X^{U,V}; \mathbb{Q}[\lambda])\right)$$
 (11.5)

which describes how families of curves on X are modified — "scattered" — as they pass through a neck modeled on  $(\mathbb{F}, V_{\infty}, V_0)$  containing the constraints  $\alpha$ .

The identity endomorphism in (11.5) is always realized as the convolution by the element

$$\mathbb{I} \in H_*(\widetilde{\mathcal{M}} \times \mathcal{H}^{V,V}_{\mathbb{F}}; \mathbb{Q}[\lambda])$$

corresponding to that part of TW coming from  $\mathbb{F}$ -trivial maps. Thus the statement that  $S_V = \mathrm{Id}$ . means that the only curves present are those which are irreducible fibers of  $\mathbb{F}$ .

**Example 11.4** When  $V = \mathbb{P}^1$ ,  $\mathbb{F} \to V$  is one of the rational ruled surfaces with its standard symplectic structure. If we wish to count all pseudo-holomorphic maps, without constraints on the genus or the induced complex structure, the relevant S-matrix is the relative TW invariant with  $(\kappa, \alpha) = (1, 1)$ . This case works out neatly: Lemma 14.6 implies that  $S_V = \mathrm{Id}$ .

**Example 11.5** When we put no constraints on either the domain or the image  $S_V$  is an operator given in terms of  $TW_{\mathbb{F}}^{V,V}$  by the S-matrix expansion (11.2). In cases where there are no rim tori in  $\mathbb{F}$ , we can expand the TW invariants in the power series (A.6) of the appendix. Letting  $TW_{\chi,A}(C_m; C_{m'})$  denote the relative invariant of  $\mathbb{F}$  satisfying the contact constraints  $C_m$  along  $V_{\infty}$  and  $C_{m'}$  along  $V_0$ , the S-matrix expansion shows that  $S_V$  has an expansion like (A.6) with coefficients

$$S_{\chi,A}(C_m; C_{m'}) = \delta_{m,m'} - TW_{\mathbb{F},\chi,A}^{V,V}(C_m; C_{m'})$$

$$+ \sum_{\substack{A_1 + A_2 = A \\ \chi_1 + \chi_2 - 2\ell(s_1) = \chi}} \sum_{m_1} \lambda^{2\ell(m_1)} \frac{|m_1|}{m_1!} TW_{\mathbb{F},\chi_1,A_1}^{V,V}(C_m; C_{m_1}) TW_{\mathbb{F},\chi_2,A_2}^{V,V}(C_{m_1^*}; C_{m'}) - \dots$$

# 12 The General Sum Formula

In all of our work thus far we have assumed that the  $(J, \nu)$ -holomorphic maps we are gluing are  $\delta$ -flat as in Definition 3.1. In this section we remove this flatness assumption and prove the symplectic sum formula in the general case.

The idea is to reduce the general case to the flat case by degenerating along many parallel copies of V. Thus instead of viewing  $Z_{\lambda}$  as the symplectic sum  $X \#_V Y$  along V we regard it as the symplectic sum of 2N+2 spaces: X and Y at the ends and 2N middle pieces each of which is a copy of the ruled space  $\mathbb F$  associated to V — see Figure 2 of the introduction. The pigeon-hole principle then implies that for large N all holomorphic maps into  $Z_{\lambda}$  are close to maps which are flat along each 'seam' of the 2N-fold sum.

**Lemma 12.1** There is a constant  $E = E_{\chi,n,A}(J,\nu)$  such that every  $(J,\nu)$ -holomorphic map into Z representing a class  $A \in H_2(Z)$  has energy at most E.

**Proof.** In an orthonormal frame  $\{e_1, e_2 = je_1\}$  on the domain, the holomorphic map equation is  $f_*e_1 + Jf_*e_2 = 2\nu(e_1)$ . Taking the norm squared and noting that  $\langle f_*e_1, Jf_*e_2 \rangle = f^*\omega(e_1, e_2)$  gives  $|df|^2 = 2|\nu|^2 + 2f^*\omega(e_1, e_2)$ . The energy is therefore the  $L^2$  norm of  $\nu$  plus the topological quantity  $\langle \omega, A \rangle$ . The lemma follows.  $\square$ 

For the remainder of this section we fix the data  $\chi, n, A, J, \nu$  which determined the constant E of Lemma 12.1 and fix an integer N with

$$N\alpha_V > E$$
 (12.1)

where  $\alpha_V < 1$  is the constant of Definition 3.1.

Fixing  $\lambda$ , we partition the neck of  $Z = Z_{\lambda}$  into 2N segments  $Z^{j}$  using the coordinate t from (2.5):

$$Z^j = \{ z \in Z_\lambda \mid (j - N - 1)\varepsilon \le t(z) \le (j - N)\varepsilon \}$$
  $j = 1, \dots, 2N$ 

where  $\varepsilon$  is as in Figure 3. Squeezing the neck at the midpoints  $t_j(z) = j - N - \frac{1}{2}$  of each of these segments defines a family

$$\mathcal{Z} \to D \subset \mathbb{C}^{2N+1} \tag{12.2}$$

as in Theorem 2.1 but with many 'necks'. Thus the fiber over  $(\mu_1, \ldots, \mu_{2N+1})$ , defined for  $|\mu| << |\lambda|$ , is a space  $Z_{\lambda}(\mu_1, \ldots, \mu_{2N+1})$  with a neck of size  $\mu_j$  inside each  $Z^j$  and the fiber over  $\mu = 0$  is the singular space obtained by connecting X to Y through a series of 2N copies of the rational ruled manifold  $\mathbb F$  associated with V. One such space is depicted in Figure 2 of the Introduction.

Fix  $\delta > 0$  such that  $\delta \leq \frac{\varepsilon}{10N}$  and consider the space  $\mathcal{M} = \mathcal{M}_{\chi,n,A}(Z_{\lambda})$  of holomorphic maps into  $Z_{\lambda}$ . Let  $f^{j}$  denote the restriction of  $f \in \mathcal{M}$  to  $f^{-1}(Z^{j})$ . We can then define an open cover of  $\mathcal{M}$  that keeps track of the values of j for which the energy  $E_{\delta}(f^{j})$  on the  $\delta$  neck around the cut is small as in equation (3.4). Specifically, to each subset  $\{i_{1}, \ldots, i_{k}\}$  of  $\{1, \ldots, 2N\}$  we associate the open subset of  $\mathcal{M}$ 

$$\mathcal{M}^{i_1,\dots i_k} = \left\{ f \in \mathcal{M} \mid E_{\delta}(f^j) < \alpha_V/2 \text{ for } j = i_1,\dots,i_k \right\}. \tag{12.3}$$

**Lemma 12.2** The  $\mathcal{M}^{i_1,...i_k}$  cover  $\mathcal{M} = \mathcal{M}_{\chi,n,A}(Z_\lambda,A)$  and set theoretically

$$\mathcal{M} = \bigcup \mathcal{M}^i - \bigcup \mathcal{M}^{i_1, i_2} + \bigcup \mathcal{M}^{i_1, i_2, i_3} - \dots$$
 (12.4)

**Proof.** Each  $f \in \mathcal{M}$  has  $\sum_{j} E(f^{j}) \leq E(f) < E$ , so (12.1) implies that  $f \in \mathcal{M}^{i}$  for at least one i. If  $E_{\delta}(f^{j}) < \alpha_{V}/2$  for exactly  $\ell$  of the j, then f is counted

$$\ell - \binom{\ell}{2} + \binom{\ell}{3} - \dots \pm \binom{\ell}{\ell} = 1$$

times on the right hand side of (12.4).

Now every  $f \in \mathcal{M}^{i_1,\dots i_k}$  has small energy in the segment  $Z^j$  for  $j = i_1, \dots, i_k$ . Replacing these  $\lambda_j$  by  $\mu_j = \mu \lambda_j$  for those values of j (and keeping the remaining  $\lambda_j$  fixed) defines a 1-parameter

subfamily  $Z_{\mu}$  of (12.2). That family degenerates in the middle of exactly k of the segments  $Z^{j}$ . At each of those degenerations f is  $\delta$ -flat in the sense of Definition 3.1. Hence

$$\mathcal{M}^{i_1,\dots i_k} = \mathcal{M}_X^V \times_{ev} (\mathcal{M}_{\mathbb{F}}^{V,V})^{k-1} \times_{ev} \mathcal{M}_Y^V$$
(12.5)

We can therefore apply the sum formula (10.6), obtaining, for a fixed A and  $\chi$ ,

$$TW_{X\#Y} = TW_X^V * \left[ \sum_{k=1}^{2N} (-1)^{k-1} \binom{2N}{k} (TW_{\mathbb{F}}^{V,V})^{k-1} \right] * TW_Y^V.$$
 (12.6)

This formula appears to be dependent on the number of cuts 2N. However, there is a way to rewrite it to see that it is independent of N. Note that after multiplying by TW the middle sum is a binomial expansion, in fact, using Lemma 11.2c,

$$\sum_{k=1}^{2N} (-1)^{k-1} \binom{2N}{k} (TW)^{k-1} = \frac{1 - (1 - TW)^{2N}}{T} = \frac{1 - (-R)^{2N}}{TW} = TW^{-1}.$$

Thus the middle part of (12.6) is exactly the S-matrix of Definition (11.3). This gives the symplectic sum formula in the general case.

**Theorem 12.3 (Symplectic Sum Formula)** Let (Z, U, W) be the symplectic sum of (X, U, V) and (Y, V, W) along V. Suppose that  $\alpha \in \mathbb{T}(Z)$  is supported off the neck as in Example 10.8. For any fixed decomposition  $(\alpha_X, \alpha_Y)$  of  $\alpha$  the relative TW invariant of Z is given in terms of the invariants of (X, U, V) and (Y, V, W) and the S-matrix (11.3) by

$$TW_Z^{U,W}(\alpha) = TW_X^{U,V}(\alpha_X) * S_V * TW_Y^{V,W}(\alpha_Y). \tag{12.7}$$

In fact, the Theorem holds more generally when  $\alpha$  separates as in Definition (10.12), except that the definition of the S-matrix needs to be enlarged. Instead of restricting  $TW_{\mathbb{F}}^{V,V}$  to  $\alpha=1$  we restrict it to the subtensor algebra  $\mathbb{T}_V$  of  $\mathbb{T}(\mathbb{F})$  generated by the kernel of the composition

$$H^*(\mathbb{F}) \xrightarrow{i^*} H^*(S_V) \xrightarrow{p_*} H^*(V)$$

where  $S_V$  is the circle bundle on  $N_V$ ,  $p_*$  is the integration along its fiber and  $i: S_V \to \mathbb{F}$  is the inclusion. In that case we get an S-matrix defined by

$$S_V = (TW_{\mathbb{F}}^{V,V}|_{\mathbb{T}_V})^{-1} \tag{12.8}$$

In the important case when U and W are empty Theorem 12.3 expresses the absolute invariant of Z in terms of the relative invariants of X and Y.

**Theorem 12.4** Let Z be the symplectic sum of (X, V) and (Y, V) and suppose that  $\alpha \in \mathbb{T}(Z)$  separates as  $(\alpha_X, \alpha_Y)$  as in Definition (10.12). Then

$$TW_Z(\alpha) = (TW_X^V * S_V * TW_Y^V)(\alpha_X, \alpha_Y). \tag{12.9}$$

where  $S_V$  is the S-matrix (12.8).

If moreover  $\alpha$  decomposes as  $\alpha = \alpha_X \otimes \alpha_Y$  then (12.9) becomes

$$TW_Z(\alpha) \ = \ TW_X^V(\alpha_X) * S_V(\alpha_V) * TW_Y^V(\alpha_Y)$$

where  $\alpha_V \in \mathbb{T}_V$  is the pullback to  $\mathbb{F}$  of the restriction of  $\alpha$  to V.

As a check, it is interesting to verify the symplectic sum formula in one very simple case where the GW invariant is simply the euler characteristic.

**Example 12.5** Consider the  $(J, \nu)$ -holomorphic maps from an elliptic curve C with fixed complex structure representing the class 0. When  $\nu = 0$  all such maps are maps to a single point, so the moduli space is X itself. Furthermore, the fiber of the obstruction bundle at a constant map p is  $H^1(T^2, p^*TX)$ , which is naturally identified with  $T_pX$ . The (virtual) moduli space for  $\nu \neq 0$  consists of the zeros of the generic section  $\overline{\nu} = \int_C \nu$  of this obstruction bundle  $TX \to X$ . Thus this particular GW invariant is  $\chi(X)$ .

Similarly, when  $\nu=0$  the moduli space of V-regular curves is  $X\setminus V$  and its V-stable compactification, defined in [IP4], is X. To compute the GW invariant relative to V, we need to know how many of these point maps become V-regular after we perturb to a generic V-compatible  $\nu\neq 0$ . Because any V-compatible  $\nu$  is tangent to V along V the corresponding section  $\overline{\nu}$  has  $\chi(X)$  zeros on X, out of which  $\chi(V)$  lie on V. Thus the relative invariant is  $GW_X^V=\chi(X)-\chi(V)$ . Note that  $\chi(\mathbb{F}_{\mathbb{V}})=2\chi(V)$ , so the S-matrix is the identity in this case. The symplectic sum formula therefore reduces to the formula

$$\chi(X) + \chi(Y) - 2\chi(V) = \chi(X \#_V Y).$$

Much more interesting examples will be given in Section 15.

Finally, can also include  $\psi$  and  $\tau$  classes as constraints. Recall that  $\phi \in H^2(\overline{\mathcal{M}}_{g,n})$  is the first chern class of  $\mathcal{L}_i$ , the relative cotangent bundle over at the *i*th marked point. There is similar bundle  $\widetilde{\mathcal{L}}_i$  over the space of stable maps whose fiber at a map f is the cotangent space to the (unstabilized) domain curve, and whose chern class is denoted by  $\psi_i$ . It is also useful to pair each  $\psi_i$  class with an  $\alpha_i \in H^*(Z)$  and consider the 'descendent'  $\tau_k(\alpha_i) = ev_i^*(\alpha_i) \cup \psi_i^k$ . It is a straightforward exercise, left to the reader, to incorporate these constraints into Theorems 12.3 and 12.4.

# 13 Constraints Passing Through the Neck

Not every constraint class  $\alpha \in H^*(Z)$  separates as in Definition 10.12. Yet for applications it is useful to have a version of the symplectic sum formula for more general constraints — ones whose Poincaré dual cuts across the neck. Since the Poincaré dual of  $\alpha \in H^*(Z)$  restricts to a class in  $H_*(X, V)$  such a general symplectic sum formula will necessarily involve relative TW invariants of classes  $\alpha \in H^*(X \setminus V)$ . That requires generalizing the relative invariant  $TW_X^V$ , which was defined in [IP4] only for constraints in  $H^*(X)$ .

We begin by recalling the 'symplectic compactification' of  $X \setminus V$  which was used in [IP4]. Let  $\hat{X}$  be the manifold obtained from  $X \setminus V$  by attaching as boundary a copy of the unit circle bundle  $p: S_V \to V$  of the normal bundle of V in X, and let  $p: \hat{X} \to X$  the natural projection. Suppose

that Z is a symplectic sum obtained by gluing  $\hat{X}$  to a similar manifold  $\hat{Y}$  along S. We can then consider stable maps in Z constrained by classes B in  $H_k(Z)$ , i.e. the set of stable maps f with the image f(x) of a marked point lying on a geometric representative of B. Restricting to the  $\hat{X}$  side, such a geometric representatives define constraints associated with classes in  $H_*(\hat{X}, S)$ .

Specifically, given a class  $B \in H_*(\hat{X}, S)$ , we can find a pseudo-manifold P with boundary Q and a map  $\phi: P \to X$  so that  $\phi(Q) \subset S$  that represents B and use this to cut-down the moduli space. Thus for generic  $(J, \nu)$ 

$$\varepsilon_V\left(\overline{\mathcal{M}}_s^V(X,A)\right)\cap p(\phi(P))$$

defines a orbifold with boundary that we denote by

$$TW_{X,A,s}^{V}(\phi). \tag{13.1}$$

After cutting down by further constraints of the appropriate dimension, this reduces to a finite set of points, giving numerical invariants constructed using  $\phi$ . This is particularly simple when  $B \in H_*(X \setminus V)$ , i.e. when B can be represented by a map into  $\hat{X} \setminus S$ . The cobordism argument of Theorem 8.1 of [IP4] then shows that the relative invariants (13.1) are well-defined. Note that these relative invariants depend on  $B \in H_*(X \setminus V)$  not on its inclusion  $B \in H_*(X)$ . For example, rim tori and the zero class in  $H_2(\hat{X}, S)$  have the same image under  $p: \hat{X} \to X$ , but might have different invariants (13.1).

In general the constrained invariant (13.1) will not be well-defined but will depend on the choice of  $\phi$ . The space

$$\mathcal{J}^V \times \text{Maps}((P,Q), (\hat{X},S))$$
 (13.2)

has a subset

$$W = \bigcup_{i=1}^n \left\{ (J, \nu, \phi) \mid \text{there is a $V$-stable } (J, \nu) \text{-holomorphic map $f$ with } f(x_i) \in p(\phi(Q)) \subset V \right\}$$

where for some map one of the marked points  $x_i$  lands on the projection of  $\phi(Q)$  into V. Except in special cases, W will have codimension one, and thus will form walls which separate (13.2) into chambers.

**Lemma 13.1** The number (13.1) is constant within a chamber. When  $B = [\phi]$  satisfies  $p_*[\partial B] = 0$  then there is only one chamber, and therefore (13.1) depends only on B.

**Proof.** Any two pairs  $(f, \phi)$  that lie in the same chamber can be connected by a path  $(f_t, \phi_t)$  with  $f_t(x_i) \in \phi_t(P \setminus Q)$ . The cobordism argument of Theorem 8.1 of [IP4] then proves the first statement.

Each B in the kernel of  $p_*\partial$  can be represented by a map  $\phi$  as above with  $\phi(Q)$  of the form  $p^{-1}(R)$  for some k-2 cycle R in V. After restricting the last factor of (13.2) to such  $\phi$ , the wall W has codimension two, giving the second statement.  $\square$ 

The following lemma relates the invariants associated with different chambers.

**Lemma 13.2** 1. If  $\phi_1, \phi_2 : P \to X$  are two maps that agree on  $\partial P$  then

$$TW_X^V(\phi_1) = TW_X^V(\phi_2) + TW_X^V(a)$$

where 
$$a = [\phi_1 \# (-\phi_2)] \in H_*(X \setminus V)$$
.

2. If  $\phi_1, \phi_2$  define the same class in  $H_*(X, V)$  then we can find  $\phi': R \to S$  where  $\partial R = Q_1 \sqcup (-Q_2)$  such that  $\phi'$  agrees with  $\phi_1$  on  $Q_1$  and agrees with  $\phi_2$  on  $Q_2$ . Then  $\phi_1$  and  $\phi_2 \# \phi'$  have the same boundary. Moreover,

$$TW_X^V(\phi_2 \# \phi') = TW_X^V(\phi_2) + TW_X^V \cdot TW_F^{VV}(\phi')$$

This actually means that in order to extend the definition of the relative invariants from [IP4], we only need to pick one geometric representative B (any one) such that  $[B] \in H_*(X, V)$ ,  $[\partial B] = \beta$  for each  $\beta \in \text{Ker } [H_{*-1}(S) \to H_{*-1}(X)]$ .

Altogether, the invariants can be thought as giving (non-canonically) a map

$$TW_X^V : \mathbb{T}(X \setminus V) \longrightarrow H_*(\mathcal{M} \times \mathcal{H}_X^V)$$
 (13.3)

although they depend on the actual representatives for the class  $\alpha$  as described in Lemma 13.2.

With this extended definition of the relative invariants the proof of Theorem 10.4 carries through. That proof began by choosing geometric representatives of constraints  $\alpha$  which separate. For a general constraint  $\alpha \in H^*(Z)$  we can still choose a geometric representative B of the Poincaré dual, and consider its restrictions  $B_X$  and  $B_Y$  to  $(\hat{X}, S)$  and  $(\hat{Y}, S)$  respectively. The remainder of the proof still applies, giving a sum formula relating the invariants  $TW_Z(\alpha)$  of Z to the relative TW invariants (13.3) of X and Y cut down by the constraints  $B_X$  and  $B_Y$ .

# 14 Relative GW Invariants in Simple cases

The symplectic sum formula of Corollary 12.4 expresses the invariants of X#Y in terms of the relative invariants of X and Y. In the next section we will apply that formula to spaces that can decomposed as symplectic sums where the spaces on one or both sides are simple enough that their relative invariants are computable. That strategy can succeed only if one has a collection of simple spaces with known relative invariants. This section provides four families of such simple spaces.

In some of the examples below the set  $\mathcal{R}$  of rim tori is non-trivial. In those cases we will give formulas for the invariants  $\overline{GW}_X^V$  defined in the appendix although, as the examples will show, it is sometimes possible to compute the  $GW_X^V$  themselves even though there are rim tori present.

#### 14.1 Riemann Surfaces

For Riemann surfaces one can consider the GW invariants as absolute invariants or relative to a finite set of points. These invariants count coverings, and the homology class A is simply the degree d of the covering.

In dimension two the symplectic sum is the same as the ordinary connect sum — one joins two Riemann surfaces by identifying a point on one with a point on another, and then smooths. Of course, to apply the sum formula one must first find  $S_V$ , which in this case is built from the relative invariants of  $(\mathbb{P}^1, V)$  where  $V = \{p_0, p_\infty\}$  two distinct points and where the constraints lie on V. In that context, we fix a nonzero degree d and two sequences s, s' that describe the multiplicities of points at the preimages of  $p_0$  and  $p_\infty$  respectively.

**Lemma 14.1** The invariants  $GW_{d,g,s,s'}^V$  with no constraints except those on  $V = \{p_0, p_\infty\}$  vanish except when g = 0 and s and s' are single points with multiplicity d. In that case

$$GW_{d,0,s,s'}^V = 1/d$$

Moreover, in dimension two the S-matrix is always the identity.

**Proof.** This invariant is the oriented count of the 0-dimensional components of  $\overline{\mathcal{M}}_{d,g,s,s'}^V$ . But using (1.15)

$$\dim \mathcal{M}_{d,q,s,s'}^{V} = 2d + 2g - 2 + \ell(s) - \deg s + \ell(s') - \deg s = 2g - 2 + \ell(s) + \ell(s')$$

is zero only if g=0 and  $\ell(s)=\ell(s')=1$ , i.e. s and s' specify single points with multiplicity d. If we stabilize, there is only one such map, given by the equation  $z\to z^d$ , so it's contribution to  $GW_{d,0,s,s'}^V$  is 1/d. This map is  $\mathbb F$ -trivial, and hence doesn't contribute to the S-matrix.  $\square$ 

The same dimension count gives the invariant with one constraint:

**Lemma 14.2** The invariants  $GW^V_{d,g,s,s'}(b)$  with one fixed branch point and no other constraints except those on  $V = \{p_0, p_\infty\}$  vanish except when g = 0 and  $\ell(s) + \ell(s') = 3$ , in which case  $GW^V_{d,0,s,s'} = 1$ .

Perhaps the most interesting two-dimensional example is the g=1 invariant of the torus  $T^2$ .

**Lemma 14.3** The g = 1 invariants of the torus relative to a set V of  $k \ge 0$  points form a series

$$GW_1^V(T^2) = \sum GW_{d,1}^V(T^2) t^d$$

that is equal to the generating function for the sum of the divisors  $\sigma(n) = \sum_{d|n} d$ , namely

$$G(t) = \sum_{n=1}^{\infty} \sigma(n) t^n = \sum_{d=1}^{\infty} \frac{dt^d}{1 - t^d}.$$
 (14.1)

**Proof.** This is a matter of counting the (unbranched) covers of the torus. That was done in [IP1] for k = 0. In general, for each degree d cover each point of V has d inverse images, each with multiplicity one. Following the notations of [IP4] we order the inverse images and divide by d!, leaving us with G(t) again.  $\Box$ 

# **14.2** $T^2 \times S^2$

Next we consider the g=1 invariants of  $X=T^2\times S^2$ . Thinking of this as an elliptic fibration over  $S^2$ , we fix a section S and two disjoint fibers F and denote the corresponding homology classes by s and f. Focusing on the classes df and s+df for  $d\geq 0$ , we can form generating functions for the absolute GW invariants and the GW invariants relative to one or two copies of the fiber.

First consider the classes df, where the invariants  $GW_{df,1}$ ,  $GW_{df,1}^F$ , and  $GW_{df,1}^{F,F}$  have dimension 0 by (1.15). There are no rim tori in  $X \setminus F$ , and when V is one or two copies of the fiber we have  $\ell = d \cdot f \cdot V = 0$ , so  $V^{\ell}$  is a point in (1.14). Therefore  $GW_{df,1}^F$  has values in  $H_2(X)$  and  $GW_{df,1}^{F,F}$  has values in  $\mathcal{H}^V = H_2(X) \times \mathcal{R}$ . Thus all three invariants can be written as power series with numerical coefficients.

Lemma 14.4 The genus one invariants GW and GW<sup>F</sup> in the classes df are given by

$$\sum_{d} GW_{df,1} t_{f}^{d} = 2G(t_{f})$$
 and  $\sum_{d} GW_{df,1}^{F} t_{f}^{d} = G(t_{f})$ 

with G(t) as in (14.1). The corresponding relative invariants  $GW^{F,F}$  are indexed by classes df + R for rim tori R and these all vanish:

$$\sum_{d} GW_{df,1}^{F,F} t_{df+R} = 0.$$

**Proof.** The generic complex structure on a topologically trivial line bundle over  $T^2$  admits no non-zero holomorphic sections. After projectivizing, we get a complex structure on  $T^2 \times S^2$  for which the only holomorphic curves representing df are multiple covers of the zero section  $F_0$  and the infinity section  $F_{\infty}$ . This is a generic V-compatible structure for  $V = F_0$  or  $F_0 \cup F_{\infty}$ . As in Lemma 14.3 these contribute G(t) to the power series for these invariants. (Note that for the relative invariant, we compute only the contribution of curves that have no components in V).

The invariants for the classes s+df are more complicated. By (1.15) the corresponding moduli spaces have dimension 4, so become points in  $\mathcal{H}_X^V$  after imposing two point constraints; these constraints can be either points  $p \in X \setminus V$ , or  $C_1(q)$ , a contact of order 1 to V at a fixed point  $q \in V$ . Again rim tori R appear only for the invariant relative to two copies of a fiber.

**Lemma 14.5** The genus one invariants GW and  $GW^F$  in the classes s + df, d > 0, are

$$\sum_{d} GW_{s+d\!f,1}(p^2) \, t_f^d \ = \ 2G'(t_f) \qquad and \qquad \sum_{d} GW_{s+d\!f,1}^F(p;C_1(p)) \, t_f^d \ = \ G'(t_f).$$

The corresponding relative invariants  $GW^{F,F}$  can be indexed by classes s + df + R for rim tori R and those with two point constraints on V vanish:

$$\sum_{d} GW_{s+df+R,1}^{F,F}(\beta) \ t_{s+df+R} = \begin{cases} 2G'(t) & \text{if } \beta = p^2 \ \text{and } R = 0, \\ G'(t) & \text{if } \beta = p; C_1(p) \ \text{and } R = 0, \\ 0 & \text{if } \beta = C_1(p); C_1(p). \end{cases}$$

**Proof.** We can compute using the product structure  $J_0$  on  $T^2 \times S^2$ . Consider a  $J_0$ -holomorphic map representing s+df, passing through generic points  $p_1$  and  $p_2$ , and whose domain is a genus 1 curve  $C = \cup C_i$ . The projection onto the second factor gives a degree 1 map  $C \to S^2$ , so C must have a rational component  $C_0$  which represents s. The projection of the remaining components is zero in homology, therefore they are multiple covers of the fibers. Because the total genus is one there is only one such component.

Summarizing, for the product structure  $J_0$  the only g=1 holomorphic curves representing s+df have two irreducible components, one of them a section S, and the other a multiple cover of a fiber  $F \notin V$ . The constraints require that S pass through  $p_1$  and F pass through  $p_2$ , or vice versa. For each of those two cases there are d choices of the marked point on the domain of F, so the count is the same as in Lemma 14.4 with G(t) replaced by G'(t). This gives the first formula.

The count for the second formula is similar. Any V-regular genus 1 holomorphic map through an interior point p and a point  $q \in V$  has two components: a section through q and a q-fold cover of a fiber q-fold through q-fold cover of a fiber q-fold formula is similar. Any q-fold cover of a fiber q-fold formula is similar. Any q-fold cover of a fiber q-fold formula is similar. Any q-fold formula is similar.

For the invariant relative two copies of the fiber F, there are rim tori, but the discussion above implies that for  $J_0$  the only holomorphic curves in the classes s + df + R appear only for R = 0 (where these curves define what R = 0 means).  $\square$ 

## 14.3 Rational Ruled Surfaces

Here let  $\mathbb{F}_n$  be the rational ruled surface whose fiber F, zero section S and infinity section E define homology classes with  $S^2 = -E^2 = n$ . We will compute some of the relative invariants  $GW^V$  with  $V = S \cup E$  and with no constraint on the complex structure of the domain  $(\kappa = 1)$ .

Fix a non-zero class A = aS + bF and two sequences s, s' of multiplicities that describe the intersection with S and E respectively. The relative GW invariant with no constraint on the complex structure and k marked points lies in the homology of the moduli space in  $X^k \times S^\ell \times E^{\ell'}$  with  $\ell = \ell(s)$  and  $\ell' = \ell(s')$ . After imposing constraints  $\alpha = (\alpha_1, \ldots, \alpha_k)$ 

$$GW_{A,g,s,s'}^{S,E}(\alpha) \in H_*(S^\ell) \otimes H_*(E^{\ell'})$$

where  $S \cong E \cong \mathbb{P}^1$ . Noting that the canonical class of  $\mathbb{F}_n$  is K = -2S + (n-2)f and  $\deg s = E \cdot A = b$  and  $\deg s' = S \cdot A = b + na$ , we have

$$\frac{1}{2}\dim GW_{A,g,s,s'}^{S,E}(\alpha) = (n+2)a + 2b + g - 1 - (\deg s - \ell(s)) - (\deg s' - \ell(s')) - \deg \alpha$$
$$= 2a + g - 1 + \ell + \ell' - \deg \alpha$$

But  $SV_s$  has dimension  $\ell(s)$ , so the moduli space represents zero in homology unless dim  $\mathcal{M}_{g,k,s,s'}(\mathbb{F}_{\kappa},A) \leq \ell + \ell'$ , so we always have

$$2a + g \le 1 + \deg \alpha. \tag{14.2}$$

**Lemma 14.6** The invariants  $GW_{A,g,s,s'}^{S,E}$  with no constraints except those on  $V = S \cup E$  vanishes except when A = bF, g = 0, and s and s' are single points with multiplicity b > 0. In that case

$$GW_{bF,0,s,s'}^{S,E} = \frac{1}{h} (S \otimes 1 + 1 \otimes E).$$

Moreover, the S-matrix in  $\mathbb{F}_n$  vanishes.

**Proof.** It suffices to show that the only contributions to GW from classes A = aS + bF come from unstable rational domains with a = 0, i.e. from  $\mathbb{F}$ -trivial maps. Taking  $\kappa = \alpha = 1$ , (14.2) implies that A = bF and g = 0 or 1. Moreover, because every bF curve intersects both E and S, we have  $\ell + \ell' \geq 2$ , and when g = 0 stability of the domain requires that  $\ell + \ell' \geq 3$ . In these cases the moduli space  $\mathcal{M}_{g,s,s'}^V(\mathbb{F},bF)$  is either empty or has dimension  $\geq 2$ .

Suppose that the moduli space is non-empty and the above stability conditions hold. Since E and S are copies of  $\mathbb{P}^1$ ,  $H_*(S^{\ell}) \otimes H_*(E^{\ell'})$  is generated by point or  $[\mathbb{P}^1]$  constraints. Then for each generic  $(J, \nu)$  there are maps f in the moduli space whose images passes through at least two fixed points  $p, q \in E \cup S$  in generic position. Take  $(J, \nu) \to (J_0, 0)$  where  $J_0$  is a complex

structure with a holomorphic projection  $\pi: \mathbb{F}_n \to \mathbb{P}^1$ . In the limit we obtain a connected stable map  $f_0$  through p and q with components representing  $a_iS + b_if$  such that  $bF = \sum a_iS + b_if$ . But then each  $a_i = 0$ , so the image of  $\pi \circ f_0$  is a single point containing  $\pi(p)$  and  $\pi(q)$ . This cannot happen for generic p,q.

Thus  $\mathcal{M}_{g,s,s'}^V(\mathbb{F},bF)$  consists of  $\mathbb{F}$ -trivial maps (c.f. Definition 11.1) representing A=bF. Such maps do not appear in the S-matrix.  $\square$ 

**Lemma 14.7** Fix a point  $p \in \mathbb{F} \setminus V$  with  $V = E \cup S$ . Then  $GW_{A,g,s,s'}^V(p)$  vanishes except in the following cases:

- (i)  $GW^{S,E}_{bF,0,s,s'}(p)=1$  when s and s' are single points with multiplicity b>0.
- (ii)  $GW_{S+bF,0.s.s'}^{S,E}(p) = SV_s \times SV_{s'}$  whenever  $\deg s = b, \ \deg s' = b + n$ .

**Proof.** From (14.2) we have  $GW^{V}_{aS+bF,g,s,s'}(p)=0$  unless  $2a+g\leq 2$ . Thus either (i) a=0, or (ii) a=1 and g=0.

In case (i) each map contributing to the invariant represents bF, passes through p, and hits E and S. Hence dim  $\mathcal{M}^V_{g,s,s'}(\mathbb{F},bF)=g-1+\ell(s)+\ell(s')$  with  $\ell(s)+\ell(s')\geq 2$ . The limiting argument used in Lemma 14.6 then shows that  $GW^V_{bF,g,s,s'}(p)$  vanishes unless g=0 and  $\ell(s)=\ell(s')=1$ . Thus s and s' are single points of multiplicity b, and the maps pass through p. Moving to the fibered complex structure, one sees that there is a unique such stable map for each b>0. This gives (i).

In case (ii) the moduli space  $\mathcal{M}_{0,s,s'}^V(\mathbb{F},S+bF)$  has dimension  $\ell(s)+\ell(s')$  and is empty unless  $b\geq 0$ . That means  $\mathcal{M}_{0,s,s'}^V(\mathbb{F},S+bF)$  is a multiple of  $SV_s\times SV_{s'}$ , so invariant vanishes except when all contact points on E and S are fixed. By the adjunction inequality, any irreducible curve C representing S+bF is rational and embedded, so we can compute the invariant by intersections in  $\mathbb{P}(H^0(\mathbb{F}_n,\mathcal{O}_{\mathbb{F}_n}(S+bF)))$  (the standard complex structure on  $\mathbb{F}_n$  is generic for these curves C because  $h^1(C;\mathcal{O}(S+bF)|_C) = h^1(\mathbb{P}^1;\mathcal{O}(n+2b)) = 0$ ). But  $h^0(\mathbb{F}_n,\mathcal{O}(S+bF)) = n+2+2b$ , and each of the conditions imposed (including multiplicities) are linear conditions. Thus the number of curves representing S+bF passing through a point p and meeting E and S at fixed contact points is 1.  $\square$ 

## 14.4 The Rational Elliptic Surface

As a final example we consider the rational elliptic surface E. Let f and f denote, respectively, the homology classes of a fiber and a fixed section of an elliptic fibration  $E \to \mathbb{P}^1$ . The following lemma describes the invariants relative to a fixed fiber F in the classes A = s + df where d is an integer. In this case there *are* rim tori in  $E \setminus F$ , suggesting that one use the average invariant  $\overline{GW}$  defined in the appendix. However, the lemma shows that the average contains only only one non-zero term (as happened in the last case of Lemma 14.5).

**Lemma 14.8** The genus g relative and absolute invariants of E in the classes  $s + df \in H_2(E)$  are related by:

$$GW_{s+df,g}(p^g) = \overline{GW}_{s+df,g}^F(p^g; C_1(f)) = GW_{s+df,g}^F(p^g; C_1(f))$$

where the second equality means that  $GW^F$  can be indexed by classes s + df + R for rim tori R and these vanish whenever  $R \neq 0$ .

**Proof.** The first equality holds because generically all maps contributing to the absolute invariant are V-regular. That is true because if some component of a stable map is taken into V = F, then that component must have genus at least 1. But then the remaining components have genus less than g, so cannot pass through g generic points.

The second equality follows from a projection argument like the one used for Lemma 14.5. Consider a curve be  $C = \bigcup C_i$  representing s + df which is holomorphic for a fibered complex structure  $J_0$  on E. Since the projection to  $\mathbb{P}^1$  gives a degree one composition, C must have a rational component  $C_0$  that intersects each fiber in exactly one point, while the other components are multiple covers of fibers, so represent  $df \in H_2(E \setminus F)$ . Moreover,  $C_0$  is an embedded section representing s. Since  $s^2 = -1$  then  $C_0$  must be the unique holomorphic curve in the class s. Thus the only curves in the class s + df + R appear only for R = 0.  $\square$ 

The invariants of Lemma 14.8 will be explicitly computed in section 15.3.

### 14.5 Rational relative invariants

Counting rational curves requires only the g=0 relative invariants and the corresponding S-matrix. The following two propositions show that these are particularly simple: the S-matrix is the identity and the relative invariant is the same as the absolute invariant in the absence of rim tori.

**Proposition 14.9** When g = 0, s = (1, ..., 1) and  $A \in H_2(X)$ , the relative invariant (summed over rim tori as in (A.1)) equals the absolute invariant:

$$\frac{1}{\ell(s)!} \overline{GW}_{A,0}^{V}(\alpha; C_s(\gamma)) = GW_{A,0}(\alpha; i_*(\gamma))$$

where  $\alpha = (\alpha_1, \dots, \alpha_n) \in (H_*(X))^n$ ,  $\gamma = (\gamma_1, \dots, \gamma_\ell) \in (H_*(V))^\ell$  and  $i_* : H_*(V) \to H_*(X)$  is the inclusion.

**Proof.** Fix a generic V-compatible pair  $(J, \nu)$ . Recall that  $(J, \nu)$  is generic for curves that have no components in V, and also its restriction to V gives a generic pair on V. However, for a curve entirely contained in V, even though  $(J, \nu)$  is generic when the curve is considered in V, it might not be generic when the curve is considered in X.

For any genus g and ordered sequence s, consider the natural inclusion:

$$\mathcal{M}^{V}_{X,A,g,s} \hookrightarrow \mathcal{M}_{X,A,g}$$

where  $A \in H_2(X)$ , so on the left we took the union over all rim tori. When s = (1, ..., 1), any element in  $\overline{\mathcal{M}}_{X,A,g}$  that has no components in V is in fact an element of  $\mathcal{M}_{X,A,g,s}^V$ . We will show that for generic V-compatible  $(J, \nu)$ , when g = 0 the contribution of the moduli space of curves with some components in V to the absolute invariant vanishes, and therefore the two invariants are equal.

For simplicity, start with the case when f has only one component, and this is entirely contained in V. Then  $A = i_*(A_0)$  with  $A_0 \in H_2(V)$ , and  $\ell(s) = A \cdot V = c_1(N_X V) \cdot A$ . Then the moduli space of such curves has

$$\dim \mathcal{M}_{V,A_0,g}(i_*\gamma) = -K_V \cdot A_0 + (\dim V - 3)(1-g) - \sum_{i=1}^{\ell} (\dim V - \dim \gamma_i)$$
$$= \dim \mathcal{M}_{X,A,g}^V(\gamma) - 1 + g$$

as in equation (6.4) of [IP4]. This means that for genus g = 0 the dimension of the moduli space of curves entirely contained in V is one less then the (virtual) dimension when considered as curves in X. Therefore if the virtual dimension in X is 0, there are no curves in V who could contribute. The general case of a curve with some components in V and some off V follows similarly.  $\square$ 

**Proposition 14.10** The g = 0 part of the S-matrix is the identity for any V and any normal bundle N.

**Proof.** By (11.3) this statement is equivalent to showing that there is no contribution to the g=0 GW-invariant coming from maps into  $\mathbb{F}$  which are not  $\mathbb{F}$ -trivial. Consider the 0 dimensional moduli space  $\mathcal{M}_{\mathbb{F},A,0,s}^{V_0,V_\infty}(\gamma)$  constrained only along  $V_0$  and  $V_\infty$ , such that the corresponding GW invariant is not zero. By Theorem 1.6 of [IP4] the same moduli space would be non-empty for the submersive structure associated with a generic  $\nu_V$  on V (as defined before Lemma 11.3). Then each  $f \in \mathcal{M}_R$  would project to a map  $f_V$  in  $\mathcal{M}_{V,\pi_*A,0,s}$  that passes through the  $\gamma$  constraints. But counting virtual dimensions using equation (6.4) of [IP4], we see that

$$\dim \mathcal{M}_{V,\pi_*A,0,s}(\gamma) = \dim \mathcal{M}_{\mathbb{F},A,0,s}^{V,V}(\gamma) - \operatorname{index} D_s^N = 0 + g - 1$$

is negative when g = 0, so this moduli space is empty for generic  $\nu_V$ .  $\Box$ 

# 15 Applications of the Sum Formula

This last section presents three applications of the sum formula: (a) the Caporaso-Harris formula for the number of nodal curves in  $\mathbb{P}^2$ , (b) the formula for the Hurwitz numbers counting branched covers of  $\mathbb{P}^1$ , and (c) the formula for the number of rational curves representing a primitive homology class in the rational elliptic surface. These formulas have all recently been established using Gromov-Witten invariants in some guise. Here we show that all three follow rather easily from the symplectic sum formula.

### 15.1 The Caporaso-Harris formula

Our first application is a derivation of the Caporaso-Harris recursion formula for the number  $N^{d,\delta}(\alpha,\beta)$  of curves in  $\mathbb{P}^2$  of degree d with  $\delta$  nodes, having a contact with L of order k at  $\alpha_k$  fixed points, and at  $\beta_k$  moving points, for  $k=1,2,\ldots$  and passing through the appropriate number r of generic fixed points in the complement of L.

For this we consider the pair  $(\mathbb{P}, L)$ , which can be written as a symplectic connect sum:

$$(\mathbb{P}^2, L) \underset{L=E}{\#} (\mathbb{F}_1, E, L) = (\mathbb{P}^2, L)$$
 (15.1)

where  $(\mathbb{F}_1, E, L)$  is the ruled surface with Euler class one with its zero section L and its infinity section E. We can then get a recursive formula for the TW invariant of  $(\mathbb{P}^2, L)$  by moving one point constraint pt to the  $\mathbb{F}$  side, and then using the symplectic sum formula.

The splitting (15.1) is along a sphere V = E = L, so there are no rim tori. The relative invariant therefore lies in the homology of SV and is invariant under the action of the subgroup

of the symmetric group that switches the order of points of same multiplicity. A basis for this homology is given by (A.4), where  $\{\gamma_i\}$  with  $\gamma^1 = p$  a point and  $\gamma_2 = [\mathbb{P}^1]$  is a basis of  $H_*(V)$ .

To recover [CH] notation, for each sequence  $(m_{a,i})$ , denote  $\alpha_a = m_{a,1}$  and  $\beta_a = m_{a,2}$ , and let  $\alpha = (\alpha_1, \alpha_2, \dots)$ ,  $\beta = (\beta_1, \beta_2, \dots)$ . Then with this change of coordinates,

$$N^{d,\delta}(\alpha,\beta) = TW_{\chi,dL,\mathbb{P}^2}^L(p^r, C_m)$$

where  $\chi - 2\delta = -d(d-3)$  is the "embedded euler characteristic" and  $r = 3d + g - 1 - \sum \alpha_i - \sum (\beta_j - 1)$ , and we are imposing no constraints on the complex structure of the curves. Similarly, let

$$N^{a,b,\chi}(\alpha',\beta'; p; \alpha,\beta) = TW_{\chi,aL+bF,\mathbb{F}}^{E,L}(C_m;p;C_{m'})$$

denote the number of curves of Euler characteristic  $\chi$  in  $\mathbb{F}$  representing aL+bF that have contact described by  $(\alpha', \beta')$  along E,  $(\alpha, \beta)$  along L and pass through an extra point  $p \in \mathbb{F}$  (we prefer to label these numbers using  $\chi$  rather than the number of nodes).

By Lemma 14.6 the S-matrix vanishes. The symplectic sum theorem then implies:

$$N^{d,\chi}(\alpha,\beta) = \sum |\alpha'| \cdot |\beta'| \cdot N^{d',\chi'}(\alpha',\beta') \cdot N^{d-d',b,\chi''}(\beta',\alpha';p;\alpha,\beta)$$

where the sum is over all  $\alpha'$ ,  $\beta'$  and all decompositions of  $(dL,\chi)$  into  $(d'L,\chi')$  and  $((d-d')L+bF,\chi'')$  such that  $\chi=\chi'+\chi''-2\ell(\alpha')-2\ell(\beta')$ . Combining Lemmas 14.6 and 14.7 we see that there are exactly two types of curves that contribute to the relative GW invariant  $TW_{\mathbb{P}^1}^{E,L}(C_{s,\gamma};p;C_{s,\gamma})$  of  $\mathbb{F}$  with one fixed point p.

1. several g = 0 unstable domain multiple covers of the fiber, one of them say of multiplicity k passing through the point p, corresponding to the situation d' = d and

$$\beta' = \beta + \varepsilon_k; \ \alpha' = \alpha - \varepsilon_k$$

where  $\varepsilon_k$  is the sequence that has a 1 in position k and 0 everywhere else.

2. several g = 0 unstable domain multiple covers of the fiber together with one g = 0 curve in the class L + aF passing through p and having all contact points with E and L fixed say described by  $\alpha'_0$  and  $\alpha_0$ ; this corresponds to d' = d - 1 and the situation

$$\alpha = \alpha_0 + \alpha'; \ \beta' = \alpha'_0 + \beta;$$
 equivalently  $\beta' \ge \beta; \ \alpha \ge \alpha'$ 

In each situation above, the number of V-stable curves is 1. In the second case, note that there are  $\binom{\alpha}{\alpha'}$  choices of  $\alpha_0$  and  $\binom{\beta'}{\beta}$  of  $\alpha'_0$ . Moreover, for each  $\mathbb{F}$ -trivial curve its invariant combines with its corresponding multiplicities in  $|s'|\ell(s')!$  to give 1. Therefore, the remaining multiplicity in case 1 is k, while in case 2, is  $|\alpha'_0| = |\beta' - \beta|$ . Putting all these together, we get:

$$N^{d,\delta}(\alpha,\beta) = \sum k N^{d,\delta'}(\alpha - \varepsilon_k, \beta + \varepsilon_k) + \sum |\beta' - \beta| \binom{\alpha}{\alpha'} \binom{\beta'}{\beta} N^{d-1,\delta'}(\alpha',\beta')$$

where the last sum is over all  $\beta' \geq \beta$ ,  $\alpha' \geq \alpha$ . This is exactly the Caporaso-Harris formula.

#### 15.2 Hurwitz numbers

The method of section 15.1 can also be applied for maps into  $\mathbb{P}^1$ . In that case the symplectic sum formula yields the cut and paste formula for Hurwitz numbers that was first proven using combinatorics by Goulden, Jackson and Vainstein in [GJV]. (Recently Li-Zhao-Zheng [LZZ] have derived a similar formula using [LR]).

The Hurwitz number  $N_{d,g}(\alpha)$  counts the number of genus g, degree d covers of  $\mathbb{P}^1$  that have the branching pattern over a fixed point  $p \in \mathbb{P}^1$  specified by the unordered partition  $\alpha$  of d, while the remaining branch points are simple and fixed. We can get at these numbers by regarding the pair  $(\mathbb{P}^1, p)$  as a symplectic sum:

$$(\mathbb{P}^1, p) = (\mathbb{P}^1, x) \underset{x=y}{\#} (\mathbb{P}^1, y, p)$$
 (15.2)

We then get a recursive formula for the GW invariant of  $(\mathbb{P}^1, p)$  by moving one simple branch point b to the  $(\mathbb{P}^1, y, p)$  side and applying the symplectic sum formula.

In fact the Hurwitz numbers are the coefficients, in a specific basis, of the GW invariants of  $\mathbb{P}^1$  relative to a point V = p. More precisely, each unordered partition  $\alpha = (\alpha_1, \alpha_2, \dots)$  of d defines numbers  $m_a = \#\{i \mid \alpha_i = a\}$ ; let  $C_m$  be the corresponding basis (A.4) (in this case the basis  $\{\gamma_i\}$  of  $H^*(V)$  has only one element). Then

$$N_{d,g}(\alpha) = GW_{\mathbb{P}^1,d,g}^p(b^r; C_m)$$

is the number of degree d, genus g covers that have the branching pattern over  $p \in \mathbb{P}^1$  determined by  $\alpha$ , and  $r = 2d - 2 + 2g - 2 - \sum (a-1)m_a$  other fixed, distinct branch points. (Note that the branching order is the order of contact to p = V). The corresponding generating function (A.6) is

$$G = GW_{\mathbb{P}^1}^p = \sum GW_{\mathbb{P}^1,d,g}^p(b^r; C_m) \prod_a \frac{(z_a)^{m_a}}{m_a!} \frac{u^r}{r!} t^d \lambda^{2g-2}.$$

Now apply the symplectic sum formula to the decomposition (15.2), putting r-1 branch points on the first copy of  $\mathbb{P}^1$  and one on the second copy. Since there are no rim tori and the S-matrix vanishes by Lemma 14.1 we obtain

$$GW_{d,q}^{p}(b^{r}; C_{m}) = \sum |m'| \cdot TW_{d,\chi_{1}}^{p}(b^{r-1}; C_{m'}) \cdot TW_{d,\chi_{2}}^{p,p}(C_{m'}; b; C_{m})$$
(15.3)

where the sum is over all  $m' = (m'_1, m'_2, ...)$  and all  $\chi_1, \chi_2$  such that  $2 - 2g = \chi_1 + \chi_2 - 2\ell(m')$  and so that the attached domain is connected. But  $TW = \exp GW$  and Lemma 14.2 implies that the only possibility for the last factor in (15.3) is a union of trivial spheres together with a degree a sphere constrained by  $C_a$  at one end and  $C_{i,j}$  with i + j = a at the other end (plus the branch point in the middle). Therefore there are only two possibilities for the other factor and for the partition  $\alpha'$  corresponding to m'.

- 1.  $\alpha=(i,j,\beta)$  and  $\alpha'=(i+j,\beta)$  for some  $i,j,\beta,$  so the covering map has genus g and degree d.
- 2.  $\alpha = (a, \beta)$  and  $\alpha' = (i, j, \beta)$  with a = i + j. Then  $\chi_1 = 2g 4$  so the covering map is either genus g 1 and degree d or genus  $g_1, g_2$  of degrees  $d_1, d_2$ .

The sum formula (15.3) can then be written as a relation for the generating function, namely

$$\partial_u G = \frac{1}{2} \sum_{i,j \ge 1} \left( ij\lambda^2 z_{i+j} \left[ \partial_{z_i} \partial_{z_j} G + \partial_{z_i} G \cdot \partial_{z_j} G \right] + (i+j)z_i z_j \partial_{z_{i+j}} G \right)$$

This is the 'cut-join' operator equation DG = 0 of [GJV]. It clearly determines the Hurwitz numbers recursively. The same formula works to give the 'Hurwitz numbers' counting branched covers of higher genus curves.

### 15.3 Curves in the rational elliptic surface

We next consider the invariants of the rational elliptic surface  $E \to \mathbb{P}^1$ . Using the notation of section 14.4 will focus on the classes A = s + df where d is an integer. The numerical invariants  $GW_{A,g}(p^g)$  then count the number of connected genus g stable maps in the class s + df through g generic points (with no constraints on the complex structure of the domain). For each g these define power series

$$F_g(t) = \sum_{d>0} GW_{d,g}(p^g) \ t^d t_s$$

where  $t = t_f$ . Recently, Bryan-Leung [BL] proved that

$$F_g(t) = F_0(t) [G'(t)]^g$$
 (15.4)

with G as in (14.1) and

$$F_0(t) = t_s \left( \prod_{d} \frac{1}{1 - t^d} \right)^{12}. \tag{15.5}$$

As mentioned in the introduction, this formula is related to the work of Yau-Zaslow [YZ] and to more general conjectures (such as those stated in [Go]) about counts of nodal curves in complex surfaces.

We will use our symplectic sum theorem to give a short proof of this formula, beginning with the g = 0 case. The proof is accomplished by relating F(t) to the similar series of elliptic (g = 1) invariants

$$H(t) = \sum_{d>0} GW_{d,1}(\tau_1[f^*]) t^d t_s$$

where  $f^* \in H^2(E)$  is the Poincaré dual of the fiber class and where  $\tau_1[f^*] = \text{ev}_1^*(f^*) \cap \psi_1$  is the corresponding 'descendent constraint' described at the end of section 12.

We will compute H in two different ways. The first is based on the standard method of 'splitting the domain', which yields the following general facts for 4-manifolds.

**Lemma 15.1** Let X be a symplectic 4-manifold with canonical class K. (a) For A = 0 and g = 1 the GW invariant with a single constraint  $B \in H^2(X)$  is

$$GW_{0,1}(B) = \frac{1}{24}K \cdot B, \tag{15.6}$$

(b) For any classes  $A, f \in H_2(X)$  satisfying  $A \cdot K = -1$ 

$$GW_{A,1}(\tau_1[f^*]) = \frac{(f \cdot A)}{24} (A^2 + K \cdot A) GW_{A,0} + \sum_{\substack{A_1 + A_2 = A \\ A_1 \neq 0, A_2 \neq 0}} (f \cdot A_2) (A_1 \cdot A_2) GW_{A_1,1} GW_{A_2,0}.$$

**Proof.** (a) For  $\nu = 0$ ,  $\overline{\mathcal{M}}_{1,1}(X,0)$  is the space  $\overline{\mathcal{M}}_{1,1} \times X$  of 'ghost tori'  $f: (T^2,j) \to X$  with f(z) = p a constant map. At such f, the fiber of the obstruction bundle is  $H^1(T^2, f^*TX) = H^1(T^2, \mathcal{O}) \otimes TX$ . The dual of the bundle  $H^1(T^2, \mathcal{O})$  over  $\overline{\mathcal{M}}_{1,1}$  is the Hodge bundle. Since the first chern number of the Hodge bundle is -1/24, the euler class the obstruction bundle is

$$\chi(X)[\overline{\mathcal{M}}_{1,1}] \otimes 1 + \frac{1}{24} 1 \otimes K \in H_2(\overline{\mathcal{M}}_{1,1} \times X).$$

For  $\nu \neq 0$ , the (virtual) moduli space is the zeros of a generic section of the obstruction bundle, which consists of (i) maps from a torus with any complex structure to  $\chi(X)$  specified points of X and (ii) maps from a torus of specified complex structure into some point on the canonical divisor. Generically, the images of the type (i) maps will miss the constraint surface representing A. The maps of type (ii) give the formula (15.6).

(b) The genus 1 topological recursive relation says

$$GW_{A,1}(\tau[f^*]) = \frac{1}{24}GW_{A,0}(H_\alpha, H^\alpha, f) + \sum_{A_1 + A_2 = A} \sum_{\alpha} GW_{A_1,1}(H_\alpha)GW_{A_2,0}(H^\alpha, f)$$

where  $\{H_{\alpha}\}$  and  $\{H^{\alpha}\}$  are bases of  $H^{*}(X)$  dual by the intersection form. But for  $A \neq 0$   $GW_{A,0}(H^{\alpha}, H^{\beta}, f)$  vanishes by dimension count unless  $H^{\alpha}$  and  $H^{\beta}$  are two-dimensional, and then each A-curve hits a generic geometric representative of  $H_{\alpha}$  at  $H^{\alpha} \cdot A$  points counted with algebraic multiplicity. A dimension count also shows that the moduli spaces with  $A_{1} = A$  and  $A_{2} = 0$  are of the wrong dimension to contribute to the double sum above. Hence the expression above becomes

$$\frac{1}{24} \sum (H^{\alpha} \cdot A)(H_{\alpha} \cdot A)(f \cdot A) GW_{A,0} + \sum_{\substack{A_1 + A_2 = A \\ A_1 \neq 0, A_2 \neq 0}} (H_{\alpha} \cdot A_1)(H^{\alpha} \cdot A_2)(f \cdot A_2) GW_{A_1,1} GW_{A_2,0}$$

plus the term with  $A_1 = 0$ , which by (15.6) is

$$\frac{1}{24} \left( K \cdot H_{\alpha} \right) GW_{A,0} \left( H^{\alpha}, f \right) = \frac{1}{24} \left( K \cdot H_{\alpha} \right) \left( A \cdot H^{\alpha} \right) \left( A \cdot f \right) GW_{A,0}.$$

The lemma follows because  $\sum (H_{\alpha} \cdot A_1)(H^{\alpha} \cdot A_2) = A_1 \cdot A_2$ .

Taking X to be the rational elliptic surface E, we can apply Lemma 15.1 with A = s + df. Then K = -f,  $A \cdot f = 1$  and  $A^2 = 2d - 1$ . The only possible decompositions are  $A_1 = kf$  and  $A_2 = s + (d - k)f$  so:

$$GW_{s+df,1}(\tau[f^*]) = \frac{d-1}{12}GW_{s+df,0} + \sum_{k=1}^{d} k \ GW_{kf,1} \ GW_{s+(d-k)f,0}$$

But for the rational elliptic surface the invariant  $GW_{kf,1}(s)$  is  $\sigma(k)$  for k > 0. (Since in  $\mathbb{P}^2$  there is a unique cubic through 9 generic points. As in section 4 of [IP1], for each k there are  $\sigma(k)$ 

distinct k-fold covers an elliptic curve with marked point, all with positive sign). Because the marked point can go to any of  $s \cdot kf = k$  points, this means that the unconstrained invariant is

$$GW_{kf,1} = \sigma(k)/k$$
 for  $k > 0$ .

It follows that

$$H(t) = \frac{1}{12} (t F_0' - F_0) + F_0 \cdot G. \tag{15.7}$$

On the other hand, we can calculate H(t) by splitting the target and using the symplectic sum theorem. Let  $\mathbb{F} = T^2 \times S^2$ , and let F denote both a fiber of the elliptic fibration E and a fixed torus  $T^2 \times \{\text{pt}\}$  inside  $\mathbb{F}$ . We can apply sum formula by writing  $E = E \#_F \mathbb{F}$  for the class A = s + df with the constraint on the  $\mathbb{F}$  side. Since  $A \cdot F = 1$ , the connected curves representing A split into the union of connected curves in E and in  $\mathbb{F}$ ; thus the symplectic sum formula applies for the GW (as well as the TW invariants).

If we have a genus 0 curve on the  $\mathbb{F}$  side in the class  $s+d_1F$ , then by projecting onto the  $T^2$  factor and noting that there are no maps from  $S^2$  to  $T^2$  of non-zero degree, we conclude that  $d_1 = 0$ . But the moduli space of genus 0 curves in  $\mathbb{F}$  representing s and passing through F is isomorphic to  $F = T^2$ , and moreover the relative cotangent bundle to them along F is isomorphic to the normal bundle to F. So

$$GW_{s,0}(\tau_1[f^*]) = GW_{s,0}((f^*)^2) = 0.$$

Thus there is no contribution from genus 0 curves on the  $\mathbb{F}$  side or in the neck (which is also a copy of  $\mathbb{F}$ ). The same argument shows that there are no rational curves in F, so the g=0 absolute and relative invariants are the same.

With these observations, the only possibility is to have a genus 1 curve on the  $\mathbb{F}$  side, genus 0 on the E side, and no contribution from the neck. The symplectic sum formula thus says

$$GW_{d,1}(\tau_1[f^*]) = \sum_{d_1+d_2=d} GW_{s+d_1f,0}(E) \cdot GW_{s+d_2f,1}(\mathbb{F})(\tau_1[f^*])$$

This last invariant can be computed by applying the topological recursive relation to  $X = \mathbb{F}$  just as in Lemma 15.1:

$$GW_{s+df,1}(\tau_1[f^*]) = \frac{d-1}{12}GW_{s+df,0} + \sum_{\substack{d_1+d_2=d\\d_1\neq 0,\ d_2\neq 0}} d_1 GW_{d_1f,1} GW_{s+d_2f,0} + d_2 GW_{s+d_1f,1} GW_{d_2f,0}.$$

But the invariants of  $\mathbb{F}$  satisfy  $GW_{df,0} = GW_{s+df,0} = 0$  for  $d \neq 0$  by the projection argument above, while for  $d \neq 0$  Lemma 14.4 gives  $d_1GW_{d_1f,1} = GW_{d_1f,1}(s) = 2\sigma(d_1)$ . We therefore get

$$H = 2F_0 \cdot \left(G - \frac{1}{24}\right). \tag{15.8}$$

Combining (15.7) with (15.8) and noting that  $F_0(0) = GW_{s,0} = 1 \cdot t_s$  we see that  $F_0$  satisfies the ODE

$$t F_0' = 12 G \cdot F_0$$

with  $F_0(0) = 1 \cdot t_s$ . Hence

$$F_0(t) = t_s \exp\left(12 \int G(t)/t \ dt\right).$$

Using the Taylor series of  $\log(1-t)$  and some elementary combinatorics, this becomes

$$F_0(t) = t_s \left( \prod_d \frac{1}{1 - t^d} \right)^{12}.$$

It remains to show (15.4) for g > 0. This case is different because for genus g > 0 the relative invariants are no longer equal to the absolute invariants. We start by fixing a fiber F of E and introducing two generating functions for the genus g relative invariant: one recording the number of curves passing through g points in  $E \setminus F$ , the other recording the number of curves passing through g - 1 points in  $E \setminus F$  plus a fixed point on F:

$$F_g^V(f) = \sum_{d} \overline{GW}_{s+df,g}^F(p^g; C_1(f))t^d,$$
  
$$F_g^V(p) = \sum_{d} \overline{GW}_{s+df,g}^F(p^{g-1}; C_1(p))t^d.$$

Using Lemma 14.8, we can relate the absolute and relative g = 1 invariants of E.

**Lemma 15.2** For X = E, the absolute and relative g = 1 invariants in the classes  $s + df \in H_2(E(1))$  are related by equations

(a) 
$$F_g = F_g^V(p) + F_{g-1}^V(f) \cdot G'$$

$$(b) \quad F_g = F_g^V(f)$$

(c) 
$$F_g = F_g(f)$$
  
(c)  $0 = F_g^V(p) \cdot F_0 + F_{g-1} \cdot F_1^V(p)$ .

**Proof.** To prove (a), we again write  $E = E \#_F \mathbb{F}$  where  $\mathbb{F} = T^2 \times S^2$ , and put g-1 points on E and the remaining point on  $\mathbb{F}$ . If we start with a class s+df the only possible decompositions are s+af and s+bf where d=a+b. Since there are g-1 points on the E side, then the genus  $g_1 \geq g-1$ . There are two possibilities:

- 1. genus g in class s + df on E and genus 0 in class s + bf on F. But that forces b = 0 so a = d.
- 2. genus g-1 in class s+df on E and genus 1 in class s+bf on  $\mathbb{F}$

Putting then together gives (a). Relation (b) is a reformulation of Lemma 14.8.

Relation (c) is seen by applying the symplectic sum formula to the sum  $K3 = E\#_F E$  (the elliptic surface K3 = E(2) is the fiber sum of E = E(1) with itself). Because a generic complex structure on K3 admits no holomorphic curves, then all relative and absolute invariants of K3 vanish. In particular, the genus g invariants through g-1 points in the class  $[s+df] \in H_2(K3)/\mathcal{R}$  vanish, where  $\mathcal{R}$  is the set of rim tori corresponding to the gluing  $K3 = E\#_F E$ .

So, for any  $g \ge 1$ , put all the g-1 points on  $X_1$  and split as above. A dimension count shows that the genus of the curve on  $X_1$  must be at least g-1, so the only possible decompositions are:

- 1. a genus g curve in the class  $s + d_1F$  on  $X_1$  and a genus 0 curve on  $X_2$  in the class  $s + d_2f$ ,  $d = d_1 + d_2$ ;
- 2. a genus g-1 curve in the class  $s+d_1F$  on  $X_1$  and a genus 1 curve on  $X_2$  in the class  $s+d_2f$ ,  $d=d_1+d_2$ ;

The symplectic sum formula then gives  $0 = F_g^V(p) \cdot F_0 + F_{g-1}^V(f) \cdot F_1^V(p)$ , which simplifies by (b).  $\square$ 

Formula (15.4) follows quickly from Lemma 15.2. Taking g=1 in Lemma 15.2d and factoring out  $F_0 \neq 0$  yields  $F_1^V(p) = 0$ . Putting that in Lemma 15.2a and again noting that  $F_0 \neq 0$  shows that  $F_g^V(p) = 0$  for all g > 0. Parts (a) and (b) of Lemma 15.2 then reduce to

$$F_a = F_{a-1} \cdot G'$$

which gives (15.4) by induction.

# 6 Appendix – Expansions of Relative TW Invariants

The Gromov-Witten invariants described in Section 1 are homology elements — the pushforward of the compactified moduli space under (1.12). These can be assembled into a power series (1.17) with coefficients in homology. Often, however, it is convenient to write the GW and TW invariants as power series whose coefficients are *numbers*, preferably numbers with clear geometric interpretations. This appendix describes how that can be done for the relative TW invariants which appear in the symplectic sum formula.

Such series expansions are easiest when we can ignore the complications caused by the covering (1.14), replacing the space  $\mathcal{H}_{X,A,s}^V$  by the more easily understood space  $V_s \cong V^{\ell(s)}$ . That can be done by pushing the homology class of the invariant down under the projection  $\varepsilon$  of (1.14), obtaining a 'summed' GW series

$$\overline{GW}_{X}^{V} = \varepsilon_{*} \left( GW_{X}^{V} \right) = \sum_{A \in H_{2}(X)} \overline{GW}_{X,A}^{V} t_{A}$$
(A.1)

whose coefficients are homology classes in  $\sqcup_s V_s$ . This is a less refined invariant, but has the advantage that its coefficients become numbers after choosing a basis of  $H^*(V)$ .

Of course (A.1) is the same as the original GW invariant when the set  $\mathcal{R}$  of (1.13) vanishes, that is, there are no rim tori. That occurs whenever  $H_1(V) = 0$  or more generally when every rim tori represents zero in  $H_2(X \setminus V)$ . We will describe the numerical expansion under that assumption; the same discussion applies to (A.1).

When there are no rim tori  $\mathcal{H}_{X,A}^V$  is the union of those  $V_s \cong V^{\ell(s)}$  with  $\deg s = A \cdot V$ . Fix a basis  $\gamma_i$  of  $H_*(V;\mathbb{Q})$ . Then a basis for the tensor algebra on  $\mathbb{N} \times H_*(V)$  is given by elements of the form

$$C_{s,I} = C_{s_1,\gamma_{i_1}} \otimes \cdots \otimes C_{s_{\ell},\gamma_{i_{\ell}}}$$
(A.2)

where  $s_i \geq 1$  are integers. Let  $\{C_{s,I}^*\}$  denote the dual basis. When  $\kappa \in H^*(\overline{\mathcal{M}})$  and  $\alpha \in \mathbb{T}(H^*(X))$ , we can expand

$$TW_X^V(\kappa,\alpha) = \sum_{s,I} \frac{1}{\ell(s)!} TW_{X,A,\chi}^V(\kappa,\alpha; C_{s,I}) C_{s,I}^* t_A \lambda^{-\chi}.$$
(A.3)

The coefficients in (A.3) have a direct geometric interpretation. Choose generic pseudomanifolds  $K \subset \overline{\mathcal{M}}_{g,n}, A_i \subset X$ , and  $\Gamma_j \subset V$  representing the Poincaré duals of  $\kappa$ ,  $\alpha$ , and the  $\gamma_j$  in their respective spaces. Then  $TW^V_{X,A,\chi}(\kappa,\alpha;C_{s,I})$  is the oriented number of genus g  $(J,\nu)$ -holomorphic, V-regular maps  $f: C \to X$  with  $C \in K$ ,  $f(x_i) \in A_i$ , and having a contact of order  $s_j$  with V along  $\Gamma_j$ . Because of that interpretation, the  $C_{s,I}$  are called "contact constraints".

While for the analysis is important to work with *ordered* sequences s, in applications it is more convenient to forget the ordering. The symmetries of the GW invariants allow us to replace the basis (A.2) with the one having elements of the form

$$\mathbf{C}_m = \prod_{a,i} \left( C_{a,\gamma_i} \right)^{m_{a,i}} \tag{A.4}$$

where  $m = (m_{a,i})$  is a finite sequence of nonnegative integers. Generalizing (1.11), we write

$$|m| = \prod_{a,i} a^{m_{a,i}}$$
  $m! = \prod_{a,i} m_{a,i}!$   $\ell(m) = \sum_{a,i} m_{a,i}$   $\deg m = \sum_{a,i} a \cdot m_{a,i}.$  (A.5)

Let  $\{z_{a,i}\}$  denote the dual basis; these generate a (super) polynomial algebra with the relations  $z_{a,i} z_{b,j} = \pm z_{b,j} z_{a,i}$  where the sign is + when  $(\deg \gamma_i)(\deg \gamma_j)$  is even. Then the generating series of the relative TW invariant is

$$TW_X^V(\kappa,\alpha) = \sum_{A,a} \sum_m TW_{X,A,\chi}^V(\kappa,\alpha; \mathbf{C}_m) \prod_{a,i} \frac{(z_{a,i})^{m_{a,i}}}{m_{a,i}!} t_A \lambda^{-\chi}$$
(A.6)

where the sum is over all sequences  $m = (m_{a,i})$  as above and where the coefficients  $TW_{X,A,\chi}^V(\kappa,\alpha; \mathbf{C}_m)$  vanish unless deg  $m = A \cdot V$ . This generating series (A.6) is formally given by

$$TW_X^V(\kappa,\alpha) = \sum_{A,g} TW_{X,A,g}^V\left(\kappa,\alpha; \exp\left(\sum_{a,i} C_{a,\gamma_i} z_{a,i}\right)\right) \ t_A \ \lambda^{-\chi}.$$

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